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Liénard–Wiechert potentials in even dimensions

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The motion of point charged particles is considered in an even dimensional Minkowski space–time. The potential functions corresponding to the massless scalar and the Maxwell fields are derived algorithmically. It is shown that in all even dimensions particles lose energy due to acceleration. [DOI: 10.1063/1.1613040]

I. INTRODUCTION

Recently Gal'tsov[1] and Kazinski *et al.*[2] have considered the Lorentz–Dirac equation for a radiating point charge in a Minkowski space–time of arbitrary dimension. They showed that the mass renormalization is possible only in three and four dimensions. In their discussion, they have also given the retarded Green's functions of the D'Alembert equation in any dimensions which was in fact constructed rigorously a long time ago.[3] Motivated by these works, we are interested in the radiation problem of accelerated point charges in all even dimensions (for the reason why we did not consider odd dimensions, please see Appendix B). Here we find the Liénard–Wiechert potentials corresponding to the massless scalar and the Maxwell fields in all even dimensions. We then use these potentials to relate the radiation from an accelerated point particle to its motion and the geometry of its trajectory. We derive the energy flux for this radiation and show that accelerating point charged particles lose energy in all even dimensions.

In Sec. II, we develop the kinematics of a curve C in a D-dimensional Minkowski manifold $\mathbf{M_D}$. In Sec. III we find the Liénard–Wiechert potentials of massless free scalar fields in an even dimensional Minkowski space. We calculate the energy radiated due to the acceleration. We show that in all even dimensions such particles lose energy, as can be expected. In Sec. IV, we determine the Liénard–Wiechert potentials for the Maxwell theory. We give a recursion relation between the vector potentials of the theory in two consecutive even dimensions. In Sec. IV, we also show that particles carrying electric charges lose energy in all even dimensions. We construct explicit solutions of the electromagnetic vector field due to the acceleration of charged particles in 4,6,8,10 dimensions. We then find the energy fluxes in 4,6,8 dimensions due to acceleration. In Appendix A, we give the Serret–Frenet equations in an arbitrary Minkowski space–time and also some auxiliary tools used in the calculation of the energy flux integrals. In Appendix B, we give a proof of the recursion relation introduced in Sec. IV.

II. CURVES IN *D*-DIMENSIONAL MINKOWSKI SPACE

In our previous works,[4–6] we developed a curve kinematics to be utilized in finding new solutions and in calculating energy fluxes due to the acceleration in the framework of Einstein's general theory of relativity. Here we use the same approach to solve the scalar and Maxwell field

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equations in all even dimensions. For this purpose, we shall now give a summary of the geometry of a regular curve in $\mathbf{M_D}$, Minkowski space–time manifold of dimension D.

Let $z^{\mu}(\tau)$ describe a smooth curve C in $\mathbf{M_D}$, where τ is the arclength parameter of the curve. From an arbitrary point x^{μ} outside the curve, there are two null lines intersecting the curve C. These points are called the retarded and the advanced times. Let Φ be the distance (world function) between the points x^{μ} and $z^{\mu}(\tau)$, then by definition it is given by

$$\Phi=\tfrac{1}{2}\,\eta_{\mu\nu}\,(x^{\mu}-z^{\mu}(\tau))\,(x^{\nu}-z^{\nu}(\tau)), \tag{1}$$

where $\eta_{\mu\nu}=\mathrm{diag}(-1,1,\ldots,1)$. Hence Φ vanishes at the retarded, τ_0, and advanced, τ_1, times. In this work we shall focus on the retarded case only. The Green's function for the vector potential chooses this point on the curve C.[7,8] By differentiating Φ with respect to x^{μ} and letting $\tau=\tau_0$, we get

$$\lambda_{\mu}\equiv\tau_{,\mu}=\frac{x_{\mu}-z_{\mu}(\tau_0)}{R},\quad R\equiv\dot{z}^{\mu}(\tau_0)\,(x_{\mu}-z_{\mu}(\tau_0)), \tag{2}$$

where R is the retarded distance, λ_{μ} is a null vector, and a dot over a letter denotes differentiation with respect to τ_0. The derivatives of R and λ_{μ}, using (2), are given by

$$\lambda_{\mu,\nu}=\frac{1}{R}[\,\eta_{\mu\nu}-\dot{z}_{\mu}\,\lambda_{\nu}-\dot{z}_{\nu}\,\lambda_{\mu}-(A-\epsilon)\,\lambda_{\mu}\lambda_{\nu}], \tag{3}$$

$$R_{,\mu}=(A-\epsilon)\,\lambda_{\mu}+\dot{z}_{\mu}, \tag{4}$$

where

$$A=\ddot{z}^{\mu}\,(x_{\mu}-z_{\mu}),\quad \dot{z}^{\mu}\,\dot{z}_{\mu}=\epsilon=0,\pm 1. \tag{5}$$

Here $\epsilon=0,-1$ for null and time-like curves, respectively. Furthermore, we have

$$\lambda_{\mu}\,\dot{z}^{\mu}=1,\quad \lambda^{\mu}\,R_{,\mu}=1. \tag{6}$$

Letting $a=A/R$, it is easy to prove that

$$a_{,\mu}\,\lambda^{\mu}=0. \tag{7}$$

Similarly, other scalars $(a_1,a_2,\ldots)$, satisfying the same property (7) obeyed by a can be defined

$$a_k\equiv\lambda_{\mu}\frac{\mathrm{d}^k\,\ddot{z}^{\mu}}{\mathrm{d}\tau_0^k},\quad k=1,2,\ldots,n. \tag{8}$$

Moreover one has

$$a_{k,\alpha}\,\lambda^{\alpha}=0, \tag{9}$$

for all k ($k=0$ is also included if we let $a_0=a$). For a more detailed discussion, please refer to Ref. 4. Here n is a positive integer which depends on the dimension D of the manifold $\mathbf{M_D}$. An analysis using Serret–Frenet frames shows that the scalars (a,a_k) are related to the curvature scalars of the curve C in $\mathbf{M_D}$. The number of such scalars is $D-1$.[9] Hence we let $n=D-1$.

III. MASSLESS SCALAR FIELD

Let ϕ describe a massless scalar field satisfying the free field equation

$$\eta^{\mu\nu}\frac{\partial^2 \phi}{\partial x^\mu \partial x^\nu}=0. \tag{10}$$

Let D be a positive even integer, and $\phi^{(D)}$ and $\phi^{(D+2)}$ denote the retarded solutions (Liénard–Wiechert potentials) of the massless scalar field in D and $D+2$ dimensions, respectively. Then

$$\phi^{(D+2)}=\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}\tau}\phi^{(D)}. \tag{11}$$

In this recursion relation we emphasize that the expressions on the right-hand side are those of D-dimensions. Take the solution $\phi^{(D)}$ in D-dimensions, take its τ derivative and divide this by the R of D-dimensions. The result is the solution $\phi^{(D+2)}$ of $D+2$-dimensions. For the proof of relation (11) see Appendix B. In the following we explicitly give these solutions for $D=4,6,8,10$:

$$\phi^{(4)}=\frac{c}{R}, \tag{12}$$

$$\phi^{(6)}=\frac{1}{R^2}[\dot{c}-pc], \tag{13}$$

$$\phi^{(8)}=\frac{1}{R^3}[\ddot{c}-3p\dot{c}+(-a_1+3p^2)c], \tag{14}$$

$$\phi^{(10)}=\frac{1}{R^4}\left[\frac{\mathrm{d}^3c}{\mathrm{d}\tau^3}-6p\ddot{c}+(15p^2-4a_1)\dot{c}+\left(-a_2+10pa_1-15p^3+\frac{1}{R}\dot{z}_\alpha\frac{\mathrm{d}^3z^\alpha}{\mathrm{d}\tau^3}\right)c\right], \tag{15}$$

where $c=c(\tau)$ is the (time dependent) scalar charge and $p\equiv a-\epsilon/R$.

The flux of massless scalar field energy is then given by (see Refs. 7 and 11 for this definition and also for the integration surface S)

$$\mathrm{d}E=-\int_S \dot{z}_\mu T_\phi^{\mu\nu}\,\mathrm{d}S_\nu, \tag{16}$$

where $T^\phi_{\mu\nu}=\partial_\mu\phi\,\partial_\nu\phi-\frac{1}{4}(\eta^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi)\eta_{\mu\nu}$ is the energy momentum tensor of the massless scalar field ϕ. The surface element $\mathrm{d}S_\mu$ on S is given by

$$\mathrm{d}S_\mu=n_\mu R^{D-3}\mathrm{d}\tau\,\mathrm{d}\Omega, \tag{17}$$

where n_ν is orthogonal to the velocity vector field $\dot{z}_\mu$ which is defined through

$$\lambda_\mu=\epsilon\dot{z}_\mu+\epsilon_1\frac{n_\mu}{R}, \qquad n^\mu n_\mu=-\epsilon R^2. \tag{18}$$

Here $\epsilon_1=\pm 1$. For the remaining part of this work we shall assume $\epsilon=-1$ (C is a time-like curve). One can consider S in the rest frame as a sphere of radius R. Here $\mathrm{d}\Omega$ is the solid angle. Letting $\mathrm{d}E/\mathrm{d}\tau=N_\phi$, we have

$$N_\phi^{(D)}=-\int_{S^{D-2}}\dot{z}_\mu T_\phi^{\mu\nu}n_\nu R^{D-3}\,\mathrm{d}\Omega, \tag{19}$$

where S^{D-2} is the $(D-2)$-dimensional sphere centered at $\tau=\tau_0$ on the curve C. At very large values of R the energy flux is given by

$$N_{\phi}^{(D)}=-\int_{S^{D-2}} d\Omega\, R^{D-3}\,(\dot{z}^{\alpha}\,\partial_{\alpha}\,\phi)(n^{\beta}\,\partial_{\beta}\,\phi).$$

It turns out that the energy flux expression has a fixed sign for all D. The energy flux of the massless scalar field ϕ as $R\rightarrow\infty$ is given by

$$N_{\phi}^{(D)}=-\epsilon_1\int_{S^{D-2}}[\xi^{(D)}]^2\, d\Omega,$$

where we obtain R independent functions (for each D) $\xi^{(D)}$ from

$$\xi^{(D)}=\lim_{R\rightarrow\infty}[R^{D/2}\,\phi^{(D+2)}].$$

As an example let $D=4$. We take $\phi^{(6)}$ from (13), multiply it by R^2 and let $R\rightarrow\infty$ (then $p\rightarrow a$), and finally we obtain $\xi^{(4)}$. The explicit expressions of $\xi^{(D)}$ are as follows:

$$\xi^{(4)}=\dot{c}-ac, \tag{20}$$

$$\xi^{(6)}=\ddot{c}-3a\dot{c}+(-a_1+3a^2)c, \tag{21}$$

$$\xi^{(8)}=\frac{d^3c}{d\tau^3}-6a\ddot{c}+(15a^2-4a_1)\dot{c}+(-a_2+10aa_1-15a^3)c, \tag{22}$$

$$\xi^{(10)}=\frac{d^4c}{d\tau^4}-10a\frac{d^3c}{d\tau^3}+(45a^2-10a_1)\ddot{c}-(5a_2-60aa_1+105a^3)\dot{c}$$
$$-(a_3-15aa_2-10a_1^2+105a_1a^2-105a^4)\,c. \tag{23}$$

Hence we have (assuming $c=$constant)

$$N_{\phi}^{(4)}=-\epsilon_1\left(\frac{4\pi}{3}\right)c^2\,\kappa_1^2, \tag{24}$$

$$N_{\phi}^{(6)}=-\epsilon_1\left(\frac{8\pi^2}{105}\right)c^2[20\kappa_1^4+7\dot{\kappa}_1^2+7\kappa_1^2\kappa_2^2], \tag{25}$$

$$N_{\phi}^{(8)}=-\epsilon_1\left(\frac{16\pi^3}{10\,395}\right)c^2\{99[(\ddot{\kappa}_1-4\kappa_1^3-\kappa_1\kappa_2^2)^2+(2\dot{\kappa}_1\kappa_2+\kappa_1\dot{\kappa}_2)^2+\kappa_1^2\kappa_2^2\kappa_3^2]$$
$$+\kappa_1^2[900\kappa_1^4+1100\kappa_1^2\kappa_2^2+3597\dot{\kappa}_1^2]\}. \tag{26}$$

IV. ELECTROMAGNETIC FIELD

In the Lorentz gauge $(\partial_{\mu}A^{\mu}=0)$, the Maxwell equations reduce to the wave equation for the vector potential A_{μ}, $\eta^{\mu\nu}\partial_{\mu}\partial_{\nu}A_{\alpha}=0$. By using the curve C, we can construct divergence free (Lorentz gauge) vector fields A_{α} satisfying the wave equation outside the curve C in any even dimension D. Similar to the case of the massless scalar field, such vectors obey the following recursion relation

$$A_{\mu}^{(D+2)}=\frac{1}{R}\frac{d}{d\tau}A_{\mu}^{(D)}. \tag{27}$$

In the recursion relation above $A_\mu^{(D)}$ is the electromagnetic vector potential in even D-dimensions, with $\mu=0,1,\ldots,D-1$. On the right-hand side of the recursion relation all operations are done in D-dimensions, just like the scalar case. However the result is to be considered as the electromagnetic vector potential of $D+2$-dimensions, with $\mu=0,1,\ldots,D+1$ on the left-hand side. As an example we have $A_\mu^{(4)}=\dot{z}_\mu/R$ as the electromagnetic vector potential of four dimensions.[8] Here $\dot{z}_\mu$ is the four velocity, R and τ are, respectively, the retarded distance and time in four dimensions. Using the recursion relation (27) the right-hand side becomes

$$\frac{\ddot{z}_\mu - a\dot{z}_\mu}{R^2} + \epsilon \frac{\dot{z}_\mu}{R^3}.$$

We then regard this expression as the solution $A_\mu^{(6)}$ of the Maxwell field equations in six-dimensions. Indeed it satisfies both the Lorentz condition and the field equations of six-dimensions, as can be verified separately. Starting from $D=4$, we can generate all even dimensional vector potentials satisfying the Maxwell equations. For instance, the vector potentials for $D=4,6,8,10$ are explicitly given by

$$A_\mu^{(4)} = \frac{\dot{z}_\mu}{R}, \tag{28}$$

$$A_\mu^{(6)} = \frac{1}{R^2}[\ddot{z}_\mu - p\dot{z}_\mu], \tag{29}$$

$$A_\mu^{(8)} = \frac{1}{R^3}\left[\frac{\mathrm{d}^3 z_\mu}{\mathrm{d}\tau^3} - 3p\ddot{z}_\mu + (-a_1+3p^2)\dot{z}_\mu\right], \tag{30}$$

$$A_\mu^{(10)} = \frac{1}{R^4}\left[\frac{\mathrm{d}^4 z_\mu}{\mathrm{d}\tau^4} - 6p\frac{\mathrm{d}^3 z_\mu}{\mathrm{d}\tau^3} + (15p^2-4a_1)\ddot{z}_\mu + \left(-a_2+10pa_1-15p^3+\frac{1}{R}\dot{z}_\alpha \frac{\mathrm{d}^3 z^\alpha}{\mathrm{d}\tau^3}\right)\dot{z}_\mu\right]. \tag{31}$$

The flux of electromagnetic energy is then given by[7] (the integration surface S is also given in this reference)

$$\mathrm{d}E = -\int_S \dot{z}_\mu T_e^{\mu\nu}\, \mathrm{d}S_\nu, \tag{32}$$

where $T^e_{\mu\nu}=F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}F^2\eta_{\mu\nu}$ is the Maxwell energy momentum tensor, $F_{\mu\nu}=A_{\nu,\mu}-A_{\mu,\nu}$ is the electromagnetic field tensor and $F^2 \equiv F^{\alpha\beta}F_{\alpha\beta}$.

Letting $\mathrm{d}E/\mathrm{d}\tau = N_e$,[10] we have

$$N_e^{(D)} = -\int_{S^{D-2}} \dot{z}_\mu T_e^{\mu\nu} n_\nu R^{D-3}\, \mathrm{d}\Omega. \tag{33}$$

At very large values of R, for all even D, we get

$$N_e^{(D)} = -\epsilon_1 \int_{S^{D-2}} \xi_\mu^{(D)} \xi_\nu^{(D)} \eta^{\mu\nu}\, \mathrm{d}\Omega, \tag{34}$$

where

$$\xi_\mu^{(D)} = \lim_{R\to\infty}[A_\mu^{(D+2)} R^{D/2}], \tag{35}$$

so that $\lambda^\mu \xi_\mu^{(D)}=0$ for all D.

Here we have two remarks. The first one is on the gauge dependence of (35). The only gauge freedom left in our solutions is $A_\mu \rightarrow A_\mu + \partial_\mu \phi$, where ϕ satisfies the scalar wave equation (10). However we have already found the solutions of the scalar wave equation for all even dimensions. It can be shown that the contribution of such scalar functions to the norm of $\xi_\mu^{(D)}$ is zero in the limit $R \rightarrow \infty$. Our second remark is on the sign of $N_e^{(D)}$ in (34). The vectors $\xi_\mu^{(D)}$ in all even dimensions are orthogonal to the null vector λ_μ, hence they must be either (i) space-like vectors, (ii) proportional to λ_μ, or (iii) zero vectors.[11] They are zero only when the curve C is a straight line which leads to no radiation. They cannot be proportional to the null vector λ_μ either, because this again leads to the trivial case of zero radiation. In the first three cases (4, 6, 8 dimensions) it can be easily observed that zero radiation implies that $\xi_\mu^{(D)}$ is a zero vector. Hence $\xi_\mu^{(D)}$ is a space-like vector in all even dimensions. Therefore the sign of the right-hand side of (34) is the same in all dimensions. These vectors are explicitly given as follows:

$$\xi_\mu^{(4)} = \ddot{z}_\mu - a \dot{z}_\mu , \tag{36}$$

$$\xi_\mu^{(6)} = \frac{d^3 z_\mu}{d\tau^3} - 3a \ddot{z}_\mu + (-a_1 + 3a^2) \dot{z}_\mu , \tag{37}$$

$$\xi_\mu^{(8)} = \frac{d^4 z_\mu}{d\tau^4} - 6a \frac{d^3 z_\mu}{d\tau^3} + (15a^2 - 4a_1) \ddot{z}_\mu + (-a_2 + 10 a a_1 - 15a^3)\, \dot{z}_\mu , \tag{38}$$

$$\begin{aligned} \xi_\mu^{(10)} = & \frac{d^5 z_\mu}{d\tau^5} - 10a \frac{d^4 z_\mu}{d\tau^4} + (45a^2 - 10a_1) \frac{d^3 z_\mu}{d\tau^3} + (-5a_2 + 60 a a_1 - 105a^3) \ddot{z}_\mu \\ & + (-a_3 + 15 a a_2 + 10 a_1^2 - 105 a_1 a^2 + 105 a^4) \dot{z}_\mu . \end{aligned} \tag{39}$$

These lead to the following energy flux expressions:

$$N_e^{(4)} = -\epsilon_1 \frac{8\pi}{3} \kappa_1^2 , \tag{40}$$

$$N_e^{(6)} = -\epsilon_1 \frac{32\pi^2}{15} \left(\dot{\kappa}_1^2 + \kappa_1^2 \kappa_2^2 + \frac{9}{7} \kappa_1^4 \right), \tag{41}$$

$$\begin{aligned} N_e^{(8)} = & -\epsilon_1 \frac{32\pi^3}{10395} \left\{ 297 \left[\left(\ddot{\kappa}_1 - \frac{4}{3} \kappa_1^3 - \kappa_1 \kappa_2^2 \right)^2 + (2 \dot{\kappa}_1 \kappa_2 + \kappa_1 \dot{\kappa}_2)^2 + \kappa_1^2 \kappa_2^2 \kappa_3^2 \right] \right. \\ & \left. + 4 \kappa_1^2 [300 \kappa_1^4 + 506 \kappa_1^2 \kappa_2^2 + 825 \dot{\kappa}_1^2] \right\} . \end{aligned} \tag{42}$$

To be compatible with the classical results,[7,8] one should take $\epsilon_1 = -1$.

V. CONCLUSION

In this work we have considered radiation of scalar and vector fields due to acceleration of point charged particles. We first examined the geometric properties of their paths in an even dimensional Minkowski space $\mathbf{M_D}$. By using the curve kinematics we developed, we have first found the retarded solutions of the scalar field equations in $\mathbf{M_D}$. These solutions describe the potentials of the accelerated scalar charges and we have examined the energy loss due to such a radiation. We have shown that in all even dimensions such scalar point particles lose energy. We have given explicit examples for $D = 4,6,8,10$. We then found the retarded solutions of the Maxwell field equations that describe the point particles carrying electric charges. Again, using the curve kinematics we developed an algorithm to calculate the vector potential A_μ in

$D+2$-dimensions from the one in D-dimensions. We have given explicit examples for $D=4,6,8$. We have calculated the energy flux in each case, and we have shown that particles lose energy due to acceleration in all even dimensions.

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APPENDIX A: SERRET–FRENET FRAMES

In this appendix, we first give the Serret–Frenet frame in D dimensions. Here we shall assume that the curve C described in Sec. II is time-like and has the tangent vector $T^{\mu}=\dot{z}^{\mu}$. Starting from this unit tangent vector, by repeated differentiation with respect to the arclength parameter τ_0, one can generate an orthonormal frame $\{T^{\mu},N_1^{\mu},N_2^{\mu},\ldots,N_{D-1}^{\mu}\}$, the *Serret–Frenet frame*:

$$\dot{T}^{\mu}=\kappa_1\, N_1^{\mu}, \tag{A1}$$

$$\dot{N}_1^{\mu}=\kappa_1\, T^{\mu}-\kappa_2\, N_2^{\mu}, \tag{A2}$$

$$\dot{N}_2^{\mu}=\kappa_2\, N_1^{\mu}-\kappa_3\, N_3^{\mu}, \tag{A3}$$

$$\cdots$$

$$\dot{N}_{D-2}^{\mu}=\kappa_{D-2}\, N_{D-3}^{\mu}-\kappa_{D-1}\, N_{D-1}^{\mu}, \tag{A4}$$

$$\dot{N}_{D-1}^{\mu}=\kappa_{D-1}\, N_{D-2}^{\mu}. \tag{A5}$$

Here κ_i $(i=1,2,\ldots,D-1)$ are the curvatures of the curve C at the point $z^{\mu}(\tau_0)$. The normal vectors N_i $(i=1,2,\ldots,D-1)$ are space-like unit vectors. Hence at the point $z^{\mu}(\tau_0)$ on the curve we have an orthonormal frame which can be used as a basis of the tangent space (of $\mathbf{M_D}$) at this point. In Sec. II, we have defined some scalars

$$a_k=\frac{\mathrm{d}^k \ddot{z}_{\mu}}{\mathrm{d}\tau_0^k}\lambda^{\mu},$$

where

$$\lambda^{\mu}=\epsilon T^{\mu}+\epsilon_1\frac{n^{\mu}}{R}.$$

Here n^{μ} is a space-like vector orthogonal to T^{μ}. It can be expressed as a linear combination of the unit vectors N_i's as

$$n^{\mu}=\alpha_1\, N_1^{\mu}+\alpha_2\, N_2^{\mu}+\cdots+\alpha_{D-1}\, N_{D-1}^{\mu},$$

where $\alpha_1^2+\alpha_2^2+\cdots+\alpha_{D-1}^2=R^2$. One can choose the spherical angles $\theta,\phi_1,\ldots,\phi_{D-4}\in(0,\pi)$, $\phi_{D-3}\in(0,2\pi)$ such that

$$\alpha_1=R\cos\theta,\quad \alpha_2=R\sin\theta\cos\phi_1,\quad \alpha_3=R\sin\theta\sin\phi_1\cos\phi_2,\ \ldots,$$

$$\alpha_{D-2}=R\sin\theta\sin\phi_1\cdots\sin\phi_{D-4}\cos\phi_{D-3},$$

$$\alpha_{D-1}=R\sin\theta\sin\phi_1\cdots\sin\phi_{D-4}\sin\phi_{D-3}.$$

Hence we can calculate the scalars a_k in terms of the curvatures of the curve C and the angles $(\theta,\phi_1,\ldots,\phi_{D-3})$ at the point $z^{\mu}(\tau_0)$. We need these expressions in the evaluation of energy flux formulas. As an example we give a and a_1:

$$a=-\epsilon\epsilon_1\kappa_1\cos\theta,\quad a_1=\kappa_1^2-\epsilon\epsilon_1\dot{\kappa}_1\cos\theta+\epsilon\epsilon_1\kappa_1\,\kappa_2\sin\theta\cos\phi_1. \tag{A6}$$

The rest of the scalars can be determined similarly. It is clear that these scalars, a_k, depend on the curvatures and the spherical angles, for all k.

APPENDIX B: THE PROOF OF THE RECURSION RELATIONS (11) AND (27)

Here we give the proof for the vector potential case. The same type of proof applies also for the scalar case. Using the recursion relation (27) successively we get

$$A_{\mu}^{(D)}=\left(\frac{1}{R}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^{(D/2)-2}\frac{\dot{z}_{\mu}}{R}. \tag{B1}$$

On the other hand, from Refs. 1 and 2, we have

$$A_{\mu}^{(D)}=\int G(x-z(\tau))\,\dot{z}_{\mu}\,\mathrm{d}\tau, \tag{B2}$$

where τ is the parameter of the curve C. The integral here is carried on the range of $\tau\in(-\infty,\infty)$. Here $G(x-z(\tau))$ is the retarded Green function given by

$$G(x-z(\tau))=\theta(x^0-z^0)\;\delta^{(D/2)-2}(\Phi). \tag{B3}$$

Here Φ is the world function given by (1), $\theta(x)$ is the Heaviside step function and $\delta^k(x)\equiv(\mathrm{d}^k/\mathrm{d}x^k)\,\delta(x)$. Here we assume that D is an even integer. [When D is an odd integer, the expression for the Green function in (B3) contains the step function instead of the δ-function. Hence the potentials in all odd dimensions remain nonlocal (integral expressions). This makes our curve kinematics ineffective.] The zeros of Φ denote the advanced and retarded proper times on the curve C, but the step function $\theta(x^0)$ chooses the retarded one. Since the integration is over the curve parameter τ in (B2), it is better to transform the derivative of the delta function with respect to Φ to the derivative with respect to τ. As a simple example consider the $D=6$ case

$$\frac{\mathrm{d}}{\mathrm{d}\Phi}\,\delta(\Phi)=\left[\frac{1}{\mathrm{d}\Phi/\mathrm{d}\tau}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,\delta(\Phi)\right]_{\Phi=0}. \tag{B4}$$

It is easy to show that $\mathrm{d}\Phi/\mathrm{d}\tau=-R$. The delta function $\delta(\Phi)$ can be expressed as follows:

$$\delta(\Phi)=\frac{\delta(\tau-\tau_0)}{R}+\frac{\delta(\tau-\tau_1)}{R}.$$

The second term will vanish identically due to the step function in (B3). Hence

$$A_{\mu}^{(6)}=\frac{1}{R}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,\frac{\dot{z}_{\mu}}{R},$$

or simply $A^{(6)}=(1/R)(\mathrm{d}/\mathrm{d}\tau)\,A^{(4)}$. This verifies our relation (27). For the general case, we need higher order derivatives of $\delta(\Phi)$ at $\Phi=0$. We find such terms by using (B4) and taking successive derivatives. In the general case, for all $k=0,1,2,\ldots$ we obtain (when $\Phi=0$)

$$\frac{\mathrm{d}^k}{\mathrm{d}\Phi^k}\,\delta(\Phi)=\left[\left(\frac{-1}{R}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^{k}\delta(\Phi)\right]. \tag{B5}$$

Using this expression in the Green's function (B3) for $k=D/2-2$, inserting it in the integral equation (B2),

$$A_{\mu}^{(D)}=\int \theta(x^0-z^0)\ \delta^{D/2-2}(\Phi)\,\dot{z}_{\mu}\,\mathrm{d}\tau \tag{B6}$$

$$=\int \theta(x^0-z^0)\left(\frac{-1}{R}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^{D/2-2}\delta(\Phi)\,\dot{z}_{\mu}\,\mathrm{d}\tau, \tag{B7}$$

and integrating by parts, we obtain (B1).

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