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On Laurent series with multiply positive coefficients

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Abstract. We consider the class of doubly infinite sequences $\{a_k\}_{k=-\infty}^{\infty}$ whose truncated sequences $\{a_k\}_{k=-n}^{n}$ are 3-times positive in the sense of Pólya and Fekete for all $n = 1, 2, \ldots$, and $a_0 \neq 0$. We obtain a characterization of this class in terms of independent parameters. We also find an estimate of the growth order of the corresponding Laurent series $\sum_{k=-\infty}^{\infty} a_k z^k$.

1. Introduction and results

Let

$$\{a_n\}_{n=-\infty}^{\infty}, \qquad a_0 \neq 0 \tag{1.1}$$

be a doubly infinite sequence, and

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \tag{1.2}$$

the corresponding generating Laurent series. Recall [5] that the sequence (1.1) is called *totally positive* if all minors of the four-way infinite matrix

$$\begin{pmatrix} & & & & \cdots & & & & \\ \cdots & a_{-2} & a_{-1} & a_0 & a_1 & a_2 & a_3 & a_4 & \cdots \\ \cdots & a_{-3} & a_{-2} & a_{-1} & a_0 & a_1 & a_2 & a_3 & \cdots \\ \cdots & a_{-4} & a_{-3} & a_{-2} & a_{-1} & a_0 & a_1 & a_2 & \cdots \\ & & & & \cdots & & & & \end{pmatrix} \tag{1.3}$$

are nonnegative. Denote by PF_∞ the class of all totally positive sequences. In 1953, Edrei [1] found an exhaustive characterization of totally positive sequences (1.1) in terms of generating functions (1.2).

Edrei's Theorem (see [1], [2, p. 427]). *A function $f(z)$ is a generating function of a totally positive sequence if and only if*

$$f(z) = Cz^k \exp(q_{-1}z^{-1} + q_1 z) \prod_{i=1}^{\infty} \frac{(1+\alpha_i z)(1+\delta_i z^{-1})}{(1-\beta_i z)(1-\gamma_i z^{-1})}\,,$$

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where k is an integer and

$$C > 0, \quad q_{-1}, q_1, \alpha_i, \beta_i, \gamma_i, \delta_i \geq 0, \quad \sum_{i=1}^{\infty}(\alpha_i + \beta_i + \gamma_i + \delta_i) < \infty.$$

In [6] Schoenberg generalized the concept of a totally positive sequence as follows: Let r be a given natural number. We say that the sequence (1.1) is *r-times positive*, provided the matrix (1.3) has no negative minor of order $\leq r$.

Denote by PF_r the class of all r–times positive sequences, $r \in \mathbf{N}$. If a Laurent series (1.2) is a generating function of an r-times positive sequence, we shall write $f \in PF_r$. Evidently, $PF_1 \supset PF_2 \supset PF_3 \supset \cdots \supset PF_\infty$. Clearly, the class PF_1 consists of all sequences (1.1) with nonnegative coefficients. It is a simple matter to see that the class PF_2 consists of all sequences of the form

$$a_n = \exp\{-\psi(n)\}, \qquad n \in \mathbf{Z}, \tag{1.4}$$

where $\psi : \mathbf{Z} \to (-\infty; +\infty]$, $\psi(0) < \infty$, is a convex function. The class PF_∞ is characterized by Edrei's Theorem. The problem of the description of the classes PF_r, $3 \leq r < \infty$, is at present far from being solved.

In view of Edrei's Theorem, Schoenberg [6] stated the problem of discovering analytical properties of the generating function (1.2) of an r-times positive sequence (1.1). He considered finite r-times positive sequences

$$(\ldots,\ 0,\ 0,\ a_0,\ a_1,\ \ldots,\ a_m,\ 0,\ 0,\ \ldots),$$

and described the zero distribution of the corresponding generating polynomials. Here we restrict ourselves to some subclasses of PF_r, $3 \leq r \leq \infty$, containing infinite sequences. Mostly, we deal with the case $r = 3$.

Denote by TQ_r, $r \in \mathbf{N} \cup \{\infty\}$, the class of all sequences (1.1) such that all truncated sequences

$$\{a_k\}_{k=-n}^{n} := \{\ldots,\ 0,\ 0,\ a_{-n},\ a_{-n+1},\ \ldots,\ a_n,\ 0,\ 0,\ \ldots\}, \quad n = 1, 2, \ldots,$$

are r-times positive. Their subclasses $Q_r \subset TQ_r$ consisting of all one-side sequences (with $a_n = 0$ for $n < 0$) were considered in [3] and [4]. We shall reduce the problem of characterization of the class TQ_3 to that of Q_3. First, we present some known facts concerning Q_3.

Theorem A (see [3]). *If a formal power series*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \tag{1.5}$$

belongs to Q_n *for some* $n \geq 3$, *then it converges on the whole complex plane* $\mathbf{C}$ *and its sum* $f(z)$ *is an entire function of order* 0. *Moreover*,

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} \leq \frac{1}{2\log\frac{1+\sqrt{5}}{2}}, \quad \text{where } M(r,f) = \max\{f(z) : |z| = r\}.$$

There is a characterization of class Q_3 in terms of independent parameters. The role of independent parameters is played by the points of the set $(0,\infty) \times [0,\infty) \times U$, where

$$U = \left\{ \{\alpha_k\}_{k=2}^{\infty} : \begin{array}{ll} \text{(i)} & 0 \leq \alpha_k \leq 1, \\ \text{(ii)} & \text{if } \exists\, j \text{ with } \alpha_j = 0, \text{ then } a_k = 0, \end{array} \quad \forall\, k \geq j \right\}. \tag{1.6}$$

Define the numbers

$$[\alpha_2] = 1 + \alpha_2, \quad [\alpha_2\alpha_3] = 1 + \alpha_3\sqrt{[\alpha_2]}, \quad [\alpha_2\alpha_3\alpha_4] = 1 + \alpha_4\sqrt{[\alpha_2\alpha_3]}, \tag{1.6$'$}$$
$$\ldots, \quad [\alpha_2\alpha_3\ldots\alpha_n] = 1 + \alpha_n\sqrt{[\alpha_2\alpha_3\ldots\alpha_{n-1}]}, \quad \ldots.$$

Theorem B (see [4]). *A power series* (1.5) *belongs to* Q_3 *if and only if*

$$a_1 = a_0\alpha, \quad a_n = \frac{a_0\alpha^n\alpha_2^{n-1}\alpha_3^{n-2}\cdots\alpha_{n-1}^2\alpha_n}{[\alpha_2]^{n/2}[\alpha_2\alpha_3]^{(n-1)/2}\cdots[\alpha_2\alpha_3\cdots\alpha_{n-1}]^{3/2}[\alpha_2\alpha_3\cdots\alpha_n]},$$

where

$$a_0 > 0, \quad \alpha \geq 0, \quad \{\alpha_k\}_{k=2}^{\infty} \in U$$

and U *is defined in* (1.6).

Since TQ_3 is a subclass of PF_2, any sequence (1.1) in TQ_3 admits representation (1.4). Set

$$N_1 = \min\{n \geq 1 : \psi(n) = +\infty\} \quad (N_1 = +\infty \text{ if } \psi(n) < +\infty,\ \forall\, n \geq 1),$$
$$N_2 = \max\{n \leq -1 : \psi(n) = +\infty\} \quad (N_2 = -\infty \text{ if } \psi(n) < +\infty,\ \forall\, n \leq -1),$$

and

$$\Delta_2\psi(n) := \begin{cases} \psi(n) - 2\psi(n-1) + \psi(n-2) & \text{if } 1 \leq n < N_1, \\ +\infty & \text{if } n \geq N_1 \text{ or } n \leq N_2, \\ \psi(n) - 2\psi(n+1) + \psi(n+2) & \text{if } N_2 < n \leq -1. \end{cases}$$

Define the sequence $\{\omega_n\}_{n=2}^{\infty}$ as follows:

$$\omega_2 = 1, \quad \omega_n = [\alpha_2\alpha_3\cdots\alpha_{n-1}]_{\alpha_2=\alpha_3=\cdots=\alpha_{n-1}=1}, \quad n \geq 3. \tag{1.7}$$

Theorem C (see [4]). *For a formal power series* (1.5) *to belong to* Q_3 *it is necessary and sufficient that*

$$\Delta_2\psi(n) \geq \log\Big(1 + \frac{1}{\sqrt{\omega_n}}\Big), \quad n \geq 2,$$

where ω_n, $n \geq 2$, *are defined in* (1.7).

Our main result allowing to reduce double-sided sequences to one-sided ones is stated in the following theorem.

Theorem 1. *A Laurent series* (1.2) *belongs to the class* TQ_3 *if and only if both power series*

$$f_1(z) = \sum_{k=0}^{\infty} a_{k-1} z^k \quad \text{and} \quad f_2(z) = \sum_{k=0}^{\infty} a_{1-k} z^k$$

belong to the class Q_3.

It is natural to ask whether the class PF_3 itself satisfies the property: both power series in Theorem 1 belong to the class PF_3. The answer is negative as the following example shows. The Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} q^{-n^2} z^n$$

belongs to PF_∞ for any $q > 1$ (see [2, p. 433]). However, by the identity

$$I_k := \begin{vmatrix} q^{-(k+1)^2} & q^{-(k+2)^2} & q^{-(k+3)^2} \\ q^{-k^2} & q^{-(k+1)^2} & q^{-(k+2)^2} \\ 0 & q^{-k^2} & q^{-(k+1)^2} \end{vmatrix} =$$

$$= q^{-(k+1)^2-(k+2)^2-(k+3)^2+6k+5}(q^6 - 2q^4 + 1), \quad k \in \mathbf{Z},$$

we conclude that $I_k < 0$ for $1 < q^2 < \frac{1+\sqrt{5}}{2}$, and hence it follows that

$$\sum_{n=k}^{\infty} q^{-n^2} z^n \notin PF_3 \quad \text{for} \quad 1 < q^2 < \frac{1+\sqrt{5}}{2} \quad \text{for any} \quad k \in \mathbf{Z}.$$

Theorems 1 and A allow us to derive the following estimates on the growth order of functions in TQ_3.

Theorem 2. *Let a Laurent series* (1.2) *belong to* TQ_r *for some* $r \geq 3$. *Then it converges in* $\mathbf{C} \setminus \{0\}$ *and*

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} \leq \frac{1}{2\log\frac{1+\sqrt{5}}{2}}, \tag{1.8}$$

$$\limsup_{r\to 0} \frac{\log M(r,f)}{(\log \frac{1}{r})^2} \leq \frac{1}{2\log\frac{1+\sqrt{5}}{2}}. \tag{1.9}$$

where $M(r,f) = \max\{f(z) : |z| = r\}$.

Combining Theorems 1 and B, we deduce a representation of the class TQ_3 in terms of independent parameters.

Theorem 3. *A Laurent series* (1.2) *belongs to* TQ_3 *if and only if*

$$a_0 = a_{-1}\alpha,$$

(1.10)
$$a_{n-1} = \frac{a_{-1}\alpha^n\alpha_2^{n-1}\alpha_3^{n-2}\cdots\alpha_{n-1}^2\alpha_n}{[\alpha_2]^{n/2}[\alpha_2\alpha_3]^{(n-1)/2}\cdots[\alpha_2\alpha_3\cdots\alpha_{n-1}]^{3/2}[\alpha_2\alpha_3\cdots\alpha_n]}, \quad n\ge 2,$$

(1.11)
$$a_{-n+1} = \frac{a_{-1}\alpha\beta^{n-1}\beta_2^{n-1}\beta_3^{n-2}\cdots\beta_{n-1}^2\beta_n}{[\beta_2]^{n/2}[\beta_2\beta_3]^{(n-1)/2}\cdots[\beta_2\beta_3\cdots\beta_{n-1}]^{3/2}[\beta_2\beta_3\cdots\beta_n]}, \quad n\ge 2,$$

where U *is defined in* (1.6) *and*

$$\alpha_{-1}>0,\ \alpha>0,\ \{\alpha_k\}_{k=2}^{\infty}\in U,\ \beta=\frac{1+\alpha_2}{\alpha\alpha_2},\ \beta_2=\alpha_2,\ \{\beta_{k+1}\}_{k=2}^{\infty}\in U.$$

Since $TQ_3\subset PF_3$, Theorem 3 provides a rich source of functions from PF_3. The important point to note here is that Theorem 3 allows us to construct functions in $PF_3\setminus PF_4$. The problem of finding the greatest $n\ge 3$, such that $f(z)\in PF_n$ for some special functions $f(z)$, was treated in ([2, Chapter 8, §12]).

Corollary. *Let* U *be defined in* (1.6). *For any* α_2, $\frac{\sqrt{5}-1}{2}<\alpha_2\le 1$, *there exist* α_3, β_3, $0<\alpha_3,\beta_3\le 1$, *such that for all* $\{\beta_{k+2}\}_{k=2}^{\infty}\in U$ *and* $\{\alpha_{k+2}\}_{k=2}^{\infty}\in U$, *the sequence* (1.1) *defined in* (1.10) *and* (1.11) *belongs to* $PF_3\setminus PF_4$.

The next theorem is an immediate consequence of Theorems 1 and C.

Theorem 4. *For a sequence* (1.4) *to belong to* TQ_3 *it is nessesary and sufficient that*

$$\Delta_2\psi(n)\ge\log\Big(1+\frac{1}{\sqrt{\omega_{n+1}}}\Big),\qquad n\ge 1$$

and

$$\Delta_2\psi(n)\ge\log\Big(1+\frac{1}{\sqrt{\omega_{-n+1}}}\Big),\quad n\le -1.$$

For a sequence (1.4) to be a PF_2-sequence, we must have the convexity of the function $\psi:\mathbf{Z}\to(-\infty;\infty]$, $\psi(0)<\infty$, that is, nonnegativity of $\Delta_2\psi(n)$. Theorem 4 demonstrates how the nonnegativity changes if we require (1.4) to belong to $TQ_3\subset PF_2$.

2. Proof of Theorem 1

Lemma 1.1. *Let a Laurent series* (1.2) *belong to* TQ_2. *Set*

$$k_1 = \min\{k > 0 : \ a_k = 0\}, \quad \textit{and} \quad k_2 = \max\{k < 0 : \ a_k = 0\}.$$

Then $a_k = 0$ *for all* $k < k_2 + 1$ *and* $k > k_1 - 1$.

Proof. Let us show that $a_k = 0$ for all $k \le k_2$. Assume $k_2 > -\infty$. We have

$$\begin{vmatrix} a_{k+1} & a_{k_2+1} \\ a_k & a_{k_2} \end{vmatrix} = -a_k a_{k_2+1} \ge 0$$

for all $k < k_2$. Hence, $a_k = 0$ for all $k \le k_2$. That $a_k = 0$ for all $k \ge k_1$ can be proved similarly. □

Without loss of generality we may assume that $a_2 \ne 0$ and $a_{-2} \ne 0$, that is, $k_1 > 2$ and $k_2 < -2$. Lemma 1.1 allows us to introduce the positive numbers

$$\delta_k = \begin{cases} a_{k-1}^2/(a_k a_{k-2}) & \text{if } \ 0 < k < k_1, \\ a_{k+1}^2/(a_k a_{k+2}) & \text{if } \ k_2 < k < 0. \end{cases}$$

Note that $\delta_1 = \delta_{-1}$.

Lemma 1.2. *Let the Laurent series* (1.2) *belong to* TQ_3. *Then both*

$$f_1(z) = \sum_{k=0}^{\infty} a_{k-1} z^k \quad \textit{and} \quad f_2(z) = \sum_{k=0}^{\infty} a_{1-k} z^k$$

belong to Q_3.

Proof. (i) Let us show that $f_1(z)$ belongs to Q_3. We shall use the following test of m-times positivity, which is due to Schonberg [6].

Theorem (see [6]). *Let* $\{b_k\}_{k=0}^{n}$ *be a finite sequence of numbers. Consider the matrices*

$$B_k = \begin{pmatrix} b_0 & b_1 & b_2 & \dots & b_n & 0 & 0 & \dots & 0 \\ 0 & b_0 & b_1 & \dots & b_{n-1} & b_n & 0 & \dots & 0 \\ 0 & 0 & b_0 & \dots & b_{n-2} & b_{n-1} & b_n & \dots & 0 \\ . & . & . & \dots & . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & . & . & . & \dots & b_n \end{pmatrix} \quad k = 1, 2, \dots, m,$$

where B_k *consists of* k *rows and* $n+k$ *columns. Assume that the following condition is satisfied for* $k = 1, 2, \dots, m$: *all* $k \times k$*–minors of* B_k *consisting of consecutive columns are strictly positive. Then* $\{\dots, 0, b_0, b_1, \dots, b_n, 0, \dots\}$ *is an* m*-times positive sequence.*

Fix any n, $1 < n < k_1$, and consider the following three matrices:

$$A_1 = \begin{pmatrix} a_{-1} & a_0 & a_1 & \dots & a_n - \varepsilon \end{pmatrix},$$

$$A_2 = \begin{pmatrix} a_{-1} & a_0 & a_1 & \dots & a_n - \varepsilon & 0 \\ 0 & a_{-1} & a_0 & \dots & a_{n-1} & a_n - \varepsilon \end{pmatrix},$$

$$A_3 = \begin{pmatrix} a_{-1} & a_0 & a_1 & \dots & a_n - \varepsilon & 0 & 0 \\ 0 & a_{-1} & a_0 & \dots & a_{n-1} & a_n - \varepsilon & 0 \\ 0 & 0 & a_{-1} & \dots & a_{n-2} & a_{n-1} & a_n - \varepsilon \end{pmatrix}.$$

All minors of A_1 are positive for $0 < \varepsilon < a_n$. For all $m \in \mathbf{N} \cap \{m \le n\}$,

$$\begin{vmatrix} a_{-m+1} & a_m & 0 \\ a_{-m} & a_{m-1} & a_m \\ 0 & a_{m-2} & a_{m-1} \end{vmatrix} = a_{-m+1}a_{m-2}a_m\Big(\delta_m - 1 - \frac{a_{m-1}a_{-m}}{a_{m-2}a_{-m+1}}\Big) \ge 0,$$

whence $\delta_m > 1$, $m \in \mathbf{N} \cap \{m \le n\}$. Therefore, all 2×2 minors of A_2 consisting of consecutive columns

$$\begin{vmatrix} a_k & a_{k+1} \\ a_{k-1} & a_k \end{vmatrix} = a_{k-1}a_{k+1}(\delta_{k+1} - 1), \qquad 0 \le k \le n-1,$$

$$\begin{vmatrix} a_{-1} & a_0 \\ 0 & a_{-1} \end{vmatrix}, \qquad \begin{vmatrix} a_n - \varepsilon & 0 \\ a_{n-1} & a_n - \varepsilon \end{vmatrix},$$

are strictly positive.

Consider all 3×3 minors of A_3 consisting of consecutive columns:

$$M_{-1} = \begin{vmatrix} a_{-1} & a_0 & a_1 \\ 0 & a_{-1} & a_0 \\ 0 & 0 & a_{-1} \end{vmatrix} > 0, \qquad M_0 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_{-1} & a_0 & a_1 \\ 0 & a_{-1} & a_0 \end{vmatrix},$$

$$M_k = \begin{vmatrix} a_k & a_{k+1} & a_{k+2}(-\varepsilon) \\ a_{k-1} & a_k & a_{k+1} \\ a_{k-2} & a_{k-1} & a_k \end{vmatrix}, \qquad 1 \le k \le n-2,$$

$$M_{n-1}(\varepsilon) = \begin{vmatrix} a_{n-1} & a_n - \varepsilon & 0 \\ a_{n-2} & a_{n-1} & a_n - \varepsilon \\ a_{n-3} & a_{n-2} & a_{n-1} \end{vmatrix},$$

$$M_n(\varepsilon) = \begin{vmatrix} a_n - \varepsilon & 0 & 0 \\ a_{n-1} & a_n - \varepsilon & 0 \\ a_{n-2} & a_{n-1} & a_n - \varepsilon \end{vmatrix} > 0.$$

Since $\{\dots, 0, 0, a_{-k-1}, a_{-k}, \dots, a_k, a_{k+1}, \dots\}$ are 3-times positive, and $\delta_k > 1$ for all $1 \le k \le n-2$, we have

$$M_0 = \begin{vmatrix} a_0 & a_1 & 0 \\ a_{-1} & a_0 & a_1 \\ 0 & a_{-1} & a_0 \end{vmatrix} + a_2 a_{-1}^2 > 0$$

and

$$M_k \geq \begin{vmatrix} a_k & a_{k+1} & a_{k+2}(-\varepsilon) \\ a_{k-1} & a_k & a_{k+1} \\ a_{k-2} & a_{k-1} & a_k \end{vmatrix} =$$

$$= \begin{vmatrix} a_k & a_{k+1} & 0 \\ a_{k-1} & a_k & a_{k+1} \\ a_{k-2} & a_{k-1} & a_k \end{vmatrix} + (a_{k+2}(-\varepsilon))a_{k-2}a_k(\delta_k - 1) > 0$$

for all sufficiently small ε.

We have

$$M_{n-1}(\varepsilon) = a_{n-1}^3 + (a_n - \varepsilon)^2 a_{n-3} - 2(a_n - \varepsilon)a_{n-1}a_{n-2},$$
$$M_{n-1}(0) \geq 0, \qquad M'_{n-1}(0) = 2a_n a_{n-3}(\delta_{n-1}\delta_n - 1) > 0.$$

Hence, $M_{n-1}(\varepsilon) > 0$ for all sufficiently small $\varepsilon > 0$. So, all 3×3 minors of A_3 consisting of consecutive columns are strictly positive. By Schoenberg's theorem stated above, it means that $\{\dots, 0, 0, a_{-1}, \dots, a_{n-1}, a_n - \varepsilon, 0, 0, \dots\}$ is a 3-times positive sequence for all sufficiently small ε. Taking the limit as ε tends to 0, we have that $\{\dots 0, 0, a_{-1}, \dots, a_{n-1}, a_n, 0, 0, \dots\}$ is a 3-times positive sequence for all $n \in \mathbf{N}$. To prove that $f_1(z) \in Q_3$, it suffices to prove that $\{\dots, 0, 0, a_{-1}, a_0, 0, 0, \dots\}$ and $\{\dots, 0, 0, a_{-1}, a_0, a_1, 0, 0, \dots\}$ are 3-times positive sequences. But the first sequence is even totally positive. And the second one is 3-times positive, that follows from $f \in TQ_3$.

(ii) Note that $f \subset TQ_3$ implies $f(1/z) \in TQ_3$. Then, by part (i), $f_2(z) \in Q_3$. □

Lemma 1.3. *Let the Laurent series*

$$f_1(z) = \sum_{k=0}^{\infty} a_{k-1} z^k \quad \text{and} \quad f_2(z) = \sum_{k=0}^{\infty} a_{1-k} z^k$$

both belong to Q_3. *Then the Laurent series* (1.2) *belongs to* TQ_3.

Proof. Fix any $n > 1$. Let k_1 and k_2 be as in Lemma 1.1. Denote

$$n_1 = \begin{cases} n & \text{if } k_1 = \infty, \\ \min\{n, k_1 - 1\} & \text{if } k_1 < \infty; \end{cases}$$

$$n_2 = \begin{cases} -n & \text{if } k_2 = -\infty, \\ \max\{-n, k_2 + 1\} & \text{if } k_2 > -\infty. \end{cases}$$

Let us prove that

$$\{\dots,\ 0,\ 0,\ a_{n_2} - \varepsilon,\ a_{n_2+1},\ \dots,\ a_{n_1-1},\ a_{n_1} - \varepsilon,\ 0,\ 0,\ \dots\}$$

is a 3-times positive sequence for any sufficiently small ε. Consider the following three matrices:

$$A_1 = \begin{pmatrix} a_{n_2} - \varepsilon & a_{n_2+1} & \dots & a_{n_1-1} & a_{n_1} - \varepsilon \end{pmatrix},$$

$$A_2 = \begin{pmatrix} a_{n_2}-\varepsilon & a_{n_2+1} & \dots & a_{n_1}-\varepsilon & 0 \\ 0 & a_{n_2}-\varepsilon & \dots & a_{n_1-1} & a_{n_1}-\varepsilon \end{pmatrix},$$

$$A_3 = \begin{pmatrix} a_{n_2}-\varepsilon & a_{n_2+1} & a_{n_2+2} & \dots & a_{n_1}-\varepsilon & 0 & 0 \\ 0 & a_{n_2}-\varepsilon & a_{n_2+1} & \dots & a_{n_1-1} & a_{n_1}-\varepsilon & 0 \\ 0 & 0 & a_{n_2}-\varepsilon & \dots & a_{n_1-2} & a_{n_1-1} & a_{n_1}-\varepsilon \end{pmatrix}.$$

All minors of A_1 are strictly positive for $0 < \varepsilon < \min\{a_{n_1}, a_{n_2}\}$. For $1 \le m \le n_1$,

$$\begin{vmatrix} a_0 & a_m & 0 \\ a_{-1} & a_{m-1} & a_m \\ 0 & a_{m-2} & a_{m-1} \end{vmatrix} = a_0 a_{m-2} a_m \Big(\delta_m - 1 - \frac{a_{-1}a_{m-1}}{a_0 a_{m-2}}\Big) \ge 0.$$

So, $\delta_m > 1$ for all m, $1 \le m \le n_1$. Similarly, $\delta_m > 1$ for all m, $n_2 \le m \le -1$. Therefore, all 2×2 minors of A_2 consisting of consecutive columns:

$$\begin{vmatrix} a_{n_2}-\varepsilon & a_{n_2+1} \\ 0 & a_{n_2}-\varepsilon \end{vmatrix}, \quad \begin{vmatrix} a_{k-1} & a_k \\ a_{k-2} & a_{k-1} \end{vmatrix} = a_k a_{k-2}(\delta_k - 1), \ \ 1 \le k \le n_1,$$

$$\begin{vmatrix} a_{k-1} & a_k \\ a_{k-2} & a_{k-1} \end{vmatrix} = a_k a_{k-2}(\delta_{k-2} - 1), \ \ n_2+2 \le k \le 1, \quad \begin{vmatrix} a_{n_1}-\varepsilon & 0 \\ a_{n_1-1} & a_{n_1}-\varepsilon \end{vmatrix}$$

are strictly positive. Consider all 3×3 minors of A_3 consisting of consecutive columns:

$$M_{n_2+1}(\varepsilon) = \begin{vmatrix} a_{n_2+1} & a_{n_2+2} & a_{n_2+3} \\ a_{n_2}-\varepsilon & a_{n_2+1} & a_{n_2+2} \\ 0 & a_{n_2}-\varepsilon & a_{n_2+1} \end{vmatrix},$$

$$M_{n_1-1}(\varepsilon) = \begin{vmatrix} a_{n_1-1} & a_{n_1}-\varepsilon & 0 \\ a_{n_1-2} & a_{n_1-1} & a_{n_1}-\varepsilon \\ a_{n_1-3} & a_{n_1-2} & a_{n_1-1} \end{vmatrix},$$

$$M_k = \begin{vmatrix} a_k & a_{k+1} & a_{k+2} \\ a_{k-1} & a_k & a_{k+1} \\ a_{k-2} & a_{k-1} & a_k \end{vmatrix}, \qquad n_2+2 \le k \le n_1-2.$$

Since $f_1(z) \in Q_3$, for $k \ge 1$ we have

$$M_k > \begin{vmatrix} a_k & a_{k+1} & 0 \\ a_{k-1} & a_k & a_{k+1} \\ a_{k-2} & a_{k-1} & a_k \end{vmatrix} \ge 0.$$

Since $f_2(z) \in Q_3$, for $k \le -1$ we have

$$M_k > \begin{vmatrix} a_k & a_{k+1} & a_{k+2} \\ a_{k-1} & a_k & a_{k+1} \\ 0 & a_{k-1} & a_k \end{vmatrix} \ge 0.$$

Since $f_1(z) \in Q_3$, we also have

$$M_0 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_{-1} & a_0 & a_1 \\ a_{-2} & a_{-1} & a_0 \end{vmatrix} > \begin{vmatrix} a_0 & a_1 & a_2 \\ a_{-1} & a_0 & a_1 \\ 0 & a_{-1} & a_0 \end{vmatrix} \geq 0.$$

The claim that $M_{n_1-1}(\varepsilon)$ and $M_{n_2+1}(\varepsilon)$ are strictly positive for all sufficiently small $\varepsilon > 0$ one can prove by the same method as in Lemma 1.2. So, all 3×3 minors of A_3 consisting of consecutive columns are strictly positive. It follows from Schoenberg's theorem, stated at the beginning of the Proof of Lemma 1.2 that

$$\{\dots,\ 0,\ 0,\ a_{n_2} - \varepsilon,\ a_{n_2+1},\ \dots,\ a_{n_1-1},\ a_{n_1} - \varepsilon,\ 0,\ 0,\ \dots\}$$

is a 3-times positive sequence for any sufficiently small $\varepsilon > 0$. Taking the limit as ε tends to 0, we find that

$$\{\dots 0,\ 0,\ a_{n_2},\ a_{n_2+1},\ \dots,\ a_{n_1-1},\ a_{n_1},\ 0,\ 0,\ \dots\}$$

is a 3-times positive sequence. □

3. Proof of Theorems 2, 3 and Corollary

Proof of Theorem 2. Consider

$$f_1(z) = \sum_{k=0}^{\infty} a_{k-1} z^k \quad \text{and} \quad f_2(z) = \sum_{k=0}^{\infty} a_{1-k} z^k.$$

By Lemma 1.2, both $f_1(z)$ and $f_2(z)$ belong to Q_3. Applying Theorem A to $f_1(z)$ and $f_2(z)$, we have

$$\limsup_{r \to \infty} \frac{\log M(r, f_i)}{(\log r)^2} \leq \frac{1}{2 \log c}, \quad c = \frac{1 + \sqrt{5}}{2},\ i = 1, 2.$$

Hence, the Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k = \frac{1}{z} f_1(z) + z f_2\Big(\frac{1}{z}\Big) - \frac{a_{-1}}{z} - a_0 - a_1 z$$

converges in $\mathbf{C} \setminus \{0\}$ and estimates (1.8) and (1.9) hold. □

Proof of Theorem 3. (i) Let $f \in TQ_3$. Then by Lemma 1.2,

$$\sum_{k=0}^{\infty} a_{k-1} z^k \in Q_3,$$

and Theorem B gives representation (1.10). Also, by Lemma 1.2,

$$\sum_{k=0}^{\infty} a_{-k+1} z^k \in Q_3$$

and Theorem B gives

$$a_0 = a_1\beta, \tag{3.1}$$

$$a_{-n+1} = \frac{a_1\beta^n\beta_2^{n-1}\beta_3^{n-2}\cdots\beta_{n-1}^2\beta_n}{[\beta_2]^{n/2}[\beta_2\beta_3]^{(n-1)/2}\cdots[\beta_2\beta_3\cdots\beta_{n-1}]^{3/2}[\beta_2\beta_3\cdots\beta_n]}, \qquad n \geq 2.$$

To prove (1.11), it remains to prove that

$$\beta = \frac{1+\alpha_2}{\alpha\alpha_2}, \qquad \beta_2 = \alpha_2.$$

The formula (1.10) for the coefficient a_1 implies

$$\frac{a_{-1}}{a_1} = \frac{1+\alpha_2}{\alpha_2\alpha^2}.$$

Formulas (1.10) and (3.1) for the coefficient a_0 give

$$a_{-1}\alpha = a_1\beta, \qquad \text{that is,} \qquad \beta = \frac{a_{-1}}{a_1}\alpha = \frac{1+\alpha_2}{\alpha_2\alpha}.$$

It follows from $\delta_1 = \delta_{-1}$ that $\alpha_2 = \beta_2$.

(ii) Consider the Laurent series

$$g_1(z) = \frac{a_{-1}}{z} + a_{-1}\alpha + \sum_{n=2}^{\infty} a_{n-1}z^{n-1},$$

where $a_{-1} > 0$, $\alpha \geq 0$ and the coefficients a_{n-1}, $n \geq 2$, are given in (1.10). By Theorem B, the sequence $\{a_k\}_{k=-1}^{\infty}$ belongs to Q_3.

Consider the Laurent series

$$g_2(z) = a_1z^{-1} + a_1\beta + \sum_{n=2}^{\infty} a_{-n+1}z^{n-1},$$

where

$$a_1 = \frac{a_{-1}\alpha^2\alpha_2}{1+\alpha_2}, \qquad \beta = \frac{1+\alpha_2}{\alpha\alpha_2}, \qquad \beta_2 = \alpha_2,$$

and the coefficients a_{-n+1}, $n \geq 2$, are given in (1.11). We have $a_1\beta = a_{-1}\alpha$ and hence,

$$a_0 = a_{-1}\alpha = a_1\beta,$$

$$a_{-n+1} = \frac{a_1\beta^n\beta_2^{n-1}\beta_3^{n-2}\cdots\beta_{n-1}^2\beta_n}{[\beta_2]^{n/2}[\beta_2\beta_3]^{(n-1)/2}\cdots[\beta_2\beta_3\cdots\beta_{n-1}]^{3/2}[\beta_2\beta_3\cdots\beta_n]}, \qquad n \geq 2.$$

By Theorem B, the sequence $\{a_{-k}\}_{k=-1}^{\infty}$ belongs to Q_3. By Theorem 1,

$$f(z) = \sum_{k=-\infty}^{\infty} a_kz^k \in TQ_3. \qquad \square$$

Proof of Corollary. For a given $\alpha > 0$ and $\frac{\sqrt{5}-1}{2} < \alpha_2 \le 1$, define the coefficients a_0, a_1, a_{-1}, a_2, a_{-2} by formulas in (1.10) and (1.11). Taking appropriate α_n and β_n, $n \ge 3$, in formulas (1.10) and (1.11), we shall define a_n, $|n| \ge 3$, such that

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k \in PF_3 \setminus PF_4.$$

Since, by (1.10) and (1.11),

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ a_{-1} & a_0 & a_1 & a_2 \\ a_{-2} & a_{-1} & a_0 & a_1 \\ a_{-3} & a_{-2} & a_{-1} & a_0 \end{vmatrix} \le \begin{vmatrix} a_0 & a_1 & a_2 & 0 \\ a_{-1} & a_0 & a_1 & a_2 \\ a_{-2} & a_{-1} & a_0 & a_1 \\ 0 & a_{-2} & a_{-1} & a_0 \end{vmatrix} =$$

$$= a_{-1}^4 \begin{vmatrix} \alpha & \frac{\alpha^2\alpha_2}{[\alpha_2]} & \frac{\alpha^3\alpha_2^2\alpha_3}{[\alpha_2]^{3/2}[\alpha_2\alpha_3]} & 0 \\ 1 & \alpha & \frac{\alpha^2\alpha_2}{[\alpha_2]} & \frac{\alpha^3\alpha_2^2\alpha_3}{[\alpha_2]^{3/2}[\alpha_2\alpha_3]} \\ \frac{\alpha\beta^2\beta_2^2\beta_3}{[\beta_2]^{3/2}[\beta_2\beta_3]} & 1 & \alpha & \frac{\alpha^2\alpha_2}{[\alpha_2]} \\ 0 & \frac{\alpha\beta^2\beta_2^2\beta_3}{[\beta_2]^{3/2}[\beta_2\beta_3]} & 1 & \alpha \end{vmatrix} =$$

$$= a_{-1}^4 \biggl(\alpha^4 - 3\frac{\alpha^4\alpha_2}{[\alpha_2]} + \frac{\alpha^4\alpha_2^2}{[\alpha_2]^2} + \frac{\alpha^8\beta^4\alpha_2^8\alpha_3^2\beta_3^2}{[\alpha_2]^6[\alpha_2\alpha_3]^2[\beta_2\beta_3]^2} +$$
$$+ 2\frac{\alpha^6\beta^2\alpha_2^4\beta_3}{[\alpha_2]^{\frac{7}{2}}[\beta_2\beta_3]} + 2\frac{\alpha^4\alpha_2^2\alpha_3}{[\alpha_2]^{\frac{3}{2}}[\alpha_2\alpha_3]} - 2\frac{\alpha^6\beta^2\alpha_2^4\beta_3\alpha_3}{[\alpha_2]^3[\alpha_2\alpha_3][\beta_2\beta_3]} -$$
$$- 2\frac{\alpha^6\beta^2\alpha_2^5\alpha_3\beta_3}{[\alpha_2]^4[\alpha_2\alpha_3][\beta_2\beta_3]} \biggr) =: a_{-1}^4\alpha^4 J(\alpha_3, \beta_3),$$

the inequality $J(\alpha_3, \beta_3) < 0$ yields $f(z) \notin PF_4$.

Substituting $\beta = (1 + \alpha_2)/(\alpha_2\alpha)$ into $J(\alpha_3, \beta_3)$, we have

$$J(\alpha_3, \beta_3) = 1 - 3\frac{\alpha_2}{[\alpha]} + \frac{\alpha_2^2}{[\alpha]^2} + \frac{\alpha_2^4\alpha_3^2\beta_3^2}{[\alpha_2]^2[\alpha_2\alpha_3]^2[\beta_2\beta_3]^2} +$$
$$+ 2\frac{\alpha_2^2\beta_3}{[\alpha_2]^{\frac{3}{2}}[\beta_2\beta_3]} + 2\frac{\alpha_2^2\alpha_3}{[\alpha_2]^{\frac{3}{2}}[\alpha_2\alpha_3]} - 2\frac{\alpha_2^2\beta_3\alpha_3}{[\alpha_2][\alpha_2\alpha_3][\beta_2\beta_3]} -$$
$$- 2\frac{\alpha_2^3\alpha_3\beta_3}{[\alpha_2]^2[\alpha_2\alpha_3][\beta_2\beta_3]} =: J(0,0) + \alpha_3 J_1 + \beta_3 J_2.$$

In all fractions entering in the last equations the numerators do not exceed 1, while the denominators are greater than 1. Therefore, $|J_1| + |J_2| \le 9$. The quantity $\alpha_3 J_1 + \beta_3 J_2$ may be made arbitrarily small by choosing sufficiently small $\alpha_3 > 0$ and $\beta_3 > 0$. Therefore, the signs of $J(\alpha_3, \beta_3)$ and $J(0,0)$ coincide for all sufficiently small $\alpha_3 > 0$ and $\beta_3 > 0$. Note that

$$J(0,0) = 1 - 3\frac{\alpha_2}{[\alpha_2]} + \frac{\alpha_2^2}{[\alpha_2]^2} = -\frac{\alpha_2^2 + \alpha_2 - 1}{[\alpha_2]^2} < 0, \quad \text{since} \quad \alpha_2 > \frac{-1+\sqrt{5}}{2}.$$

$\square$

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О рядах Лорана с многократно положительными коэффициентами

Н. А. ЖЕЛТУКИНА

Рассмотрен класс последовательностей $\{a_k\}_{k=-\infty}^{\infty}$, отрезки которых $\{a_k\}_{k=-n}^{n}$ являются 3-кратно положительными в смысле Полиа–Фекете при всех $n = 1, 2, \dots$, и $a_0 \neq 0$. Получено описание этого класса в терминах независимых параметров. Найдены оценки роста соответствующих рядов Лорана $\sum_{k=-\infty}^{\infty} a_k z^k$.

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