

LETTER TO THE EDITOR

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LETTER TO THE EDITOR

Single-particle entanglement

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Online at stacks.iop.org/JOptB/7/L1**Abstract**

Using the approach to quantum entanglement based on the quantum fluctuations of observables, we show the existence of perfect entangled states of a single ‘spin-1’ particle. We give physical examples related to photons and condensed matter physics.

Keywords: entanglement, qutrit, biphoton, spin-1

In the usual treatment, quantum entanglement is associated with the specific *nonlocal* correlations among the parts of a quantum system that has no classical analogue (e.g., see [1]). This assumes that the entangled system should consist of two or more parts. At the same time, there is a strong interest in single-particle (especially single-photon) entanglement [2–8]. In particular, the possibility to use single-photon entanglement in quantum cryptography has been discussed recently [7].

Single-photon entanglement is usually considered in terms of two-qubit entanglement. One of the qubits is an intrinsic property of the photon like polarization, while the second qubit corresponds to the spatial degrees of freedom, defined by the two spatial modes of a single photon. These modes can be produced either by a beam splitter [2, 3, 8] or through the use of two identical cavities, containing a single excitation [6].

Undoubtedly, it is of high interest to consider entanglement caused only by the intrinsic degrees of freedom of a single particle.

Here we examine single-particle entanglement from the perspective of a recent approach, treating entanglement as a manifestation of quantum fluctuations in a state where the fluctuations come to their extreme [9–13]. In particular, it was shown that the *completely entangled* (CE) states of a given system can be defined in terms of a certain variational principle for the total amount of quantum fluctuation [12]. It should be stressed that every entangled state can be transformed into a CE state by an *SLOCC* transformation [14–16] that can change the amount of entanglement but cannot either create or destroy it. Mathematically *SLOCC* transformation amounts to action of the *complexified* dynamic group G^c .

The essence of the approach can be formulated as follows [11, 12, 17]. Let \mathbb{H}_A be the space of states of a quantum

system A, and \mathcal{L} be the Lie algebra generated by observables we are going to measure in the course of experiment, or, which is the same, by the Hamiltonians available for manipulation with quantum states. \mathcal{L} is said to be the Lie algebra of *essential observables*, and the corresponding compact group $G = \exp(\mathcal{L})$ is called the *dynamic group* of system A. For example, for a two-component system $\mathbb{H}_{AB} = \mathbb{H}_A \otimes \mathbb{H}_B$ with full access to local degrees of freedom the dynamic group is $SU(\mathbb{H}_A) \times SU(\mathbb{H}_B)$. The corresponding group of *SLOCC* transformations $G^c = \exp(\mathcal{L}^c)$ is defined by complexified algebra $\mathcal{L}^c = \mathcal{L} \otimes \mathbb{C}$. In the above example, $G^c = SL(\mathbb{H}_A) \times SL(\mathbb{H}_B)$.

The key physical quantity responsible for entanglement of a state $\psi \in \mathbb{H}_A$ is its *total variation*

$$V_{\text{tot}}(\psi) = \sum_i (\langle \psi | \mathcal{O}_i^2 | \psi \rangle - \langle \psi | \mathcal{O}_i | \psi \rangle^2), \quad (1)$$

where summation is performed over an orthonormal basis \mathcal{O}_i of the Lie algebra of essential observables \mathcal{L} . The crucial point is that this quantity is independent of the basis \mathcal{O}_i , and reflects the total *amount of quantum fluctuation* of the system in the state ψ . For spin group $SU(2)$, one can use spin projection operators S_x, S_y, S_z as the basis of $\mathcal{L} = su(2)$.

The quantity (1) bears a similarity with the so-called *skew information* that has been introduced by Wigner [18, 19] to specify the amount of information, carried by a quantum state with respect to noncommuting observables, whose measurement needs macroscopic apparatus. In turn, the observables associated with the additive conserved quantities like energy can be measured with microscopic apparatus. The main difference between our approach and that of Wigner

consists in the definition of fundamental observables in terms of the dynamic symmetry of the system.

To clarify the physical meaning of (1), note that in the case of classical observables represented by c -numbers the total amount of fluctuation is equal to zero. Thus, the nonzero value of (1) specifies the *remoteness* of the state ψ from the ‘classical reality’, i.e., from the result of classical measurements.

CE states $\psi_{\text{CE}} \in \mathbb{H}_S$ have the following *extremality property* [12]:

$$V_{\text{tot}}(\psi_{\text{CE}}) = \max_{\psi \in \mathbb{H}_S} V_{\text{tot}}(\psi). \quad (2)$$

This means that CE states provide the maximal amount of quantum fluctuation in the system. In other words, CE states are maximally remote from the ‘classical reality’. This clarifies the fact that entanglement has no classical analogue.

In contrast, generalized coherent states correspond to the minimal amount of quantum fluctuation [17] (concerning generalized coherent states, see [20]). Thus, they are closest to the ‘classical reality’.

Equation (2) plays in entanglement the same role as variational principles in mechanics. Using a differential criterion of extremum, one can recast it into the form

$$\langle \psi_{\text{CE}} | \mathcal{O} | \psi_{\text{CE}} \rangle = 0, \quad \forall \mathcal{O} \in \mathcal{L}, \quad (3)$$

which tells us that in the CE state the system is in the centre of its quantum fluctuations. The definition (3) does not assume the nonlocality of system A, and therefore can be used to study entanglement in single-component systems.

As an example of some practical interest consider a spin-1 system with dynamic group $SU(2)$ in its three-dimensional irreducible representation \mathbb{H}_1 . An example is provided by a single photon with orbital angular momentum $l = 1$ [21, 22]. Another example is given by an electric dipole photon with total angular momentum $j = 1$ [23, 24]. One more realization is provided by the superfluid ^3He , where both spin and orbital momenta of a Cooper pair are equal to one [25, 26].

To clarify the structure of CE states in a single spin-1 system we start with the Clebsch–Gordon decomposition

$$\mathbb{H}_{1/2} \otimes \mathbb{H}_{1/2} = \mathbb{H}_1 \oplus \mathbb{H}_0, \quad (4)$$

of two spin- $\frac{1}{2}$ systems into a symmetric component \mathbb{H}_1 of spin 1, and a skew symmetric scalar component \mathbb{H}_0 . If we denote the base states in $\mathbb{H}_{1/2}$ by $|\uparrow\rangle$ and $|\downarrow\rangle$, then the basis of \mathbb{H}_1 is represented by the symmetric triplet

$$\begin{aligned} & |\uparrow\uparrow\rangle \\ & |\downarrow\downarrow\rangle \\ & \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \end{aligned} \quad (5)$$

while the antisymmetric singlet

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \quad (6)$$

corresponds to \mathbb{H}_0 . Since the states of the spin-1 system under consideration can always be specified by the projection of spin onto the quantization axis $|m\rangle$, the states (5) can be interpreted

as the states $|m = 1\rangle$, $|m = -1\rangle$, and $|m = 0\rangle$, respectively. From the physical point of view, this means that if a single spin-1 system, prepared initially in the state $|m = 0\rangle$, decays into the two spin- $\frac{1}{2}$ objects, they should be observed in the EPR (Einstein–Podolsky–Rosen) state (the last state in (5)). This is an indication that the spin-1 state $|m = 0\rangle$ is entangled. The other two states $|m = \pm 1\rangle$ in the triplet (5) are coherent and decay into disentangled spin- $\frac{1}{2}$ components.

To classify spin-1 states, it is convenient to treat \mathbb{H}_1 as a complexification of three-dimensional Euclidean space

$$\mathbb{H}_1 = \mathbb{E}^3 \otimes \mathbb{C} \quad (7)$$

with dynamical symmetry group $SU(2) \approx SO(3)$, acting by rotations in \mathbb{E}^3 . Then, every state $|\psi\rangle$ in \mathbb{H}_1 can be represented as the complex superposition

$$|\psi\rangle = \cos \varphi \cdot |\vec{\mu}\rangle + i \sin \varphi \cdot |\vec{\nu}\rangle, \quad 0 \leq \varphi \leq \pi/4, \quad (8)$$

of two orthonormal vectors $\vec{\mu}, \vec{\nu} \in \mathbb{E}^3$. Note that one orthonormal pair $\vec{\mu}, \vec{\nu} \in \mathbb{E}^3$ can be transformed into another by a rotation. Hence, the angle φ is the unique *intrinsic* invariant of the spin-1 state. Therefore, it is not surprising that its measure of entanglement can be expressed via φ .

We will see later that $\varphi = 0$ corresponds to the CE states, while $\varphi = \pi/4$ gives unentangled (coherent) states. In the theory of superfluid ^3He , the former are known as the *unitary* states.

Spin projection operator $S_{\vec{\omega}}$ onto direction $\vec{\omega} \in \mathbb{E}^3$ in representation (7) amounts to infinitesimal rotation with angular velocity $\vec{\omega}$ given by the cross product

$$S_{\vec{\omega}} : x \mapsto i\vec{\omega} \times \vec{x}, \quad \vec{x} \in \mathbb{E}^3. \quad (9)$$

Hence, $S_{\vec{v}}|\vec{v}\rangle = 0$, i.e. $|\vec{v}\rangle$ is a state with zero spin projection onto direction \vec{v} . Moreover, by (9),

$$\langle \vec{v} | S_{\vec{\omega}} | \vec{v} \rangle = (\vec{v}, \vec{\omega}, \vec{v}) = 0, \quad \forall \vec{\omega} \in \mathbb{E}^3 \quad (10)$$

and by criterion (3), $|\vec{v}\rangle$ is the CE state. For the general state (8), we get

$$\langle \psi | S_{\vec{\omega}} | \psi \rangle = 2 \sin \varphi \cos \varphi (\vec{\mu}, \vec{\omega}, \vec{\nu}) = \sin(2\varphi) (\vec{\mu}, \vec{\omega}, \vec{\nu}). \quad (11)$$

Hence, $|\psi\rangle$ is the CE state only for $\varphi = 0$. So, we arrive at a characterization of CE states as those with spin projection $m = 0$ onto some direction. Typical examples are the states

$$\begin{aligned} |\psi_0\rangle &= |0\rangle \\ |\psi_{\pm 1}\rangle &= \frac{1}{\sqrt{2}}(|+1\rangle \pm |-1\rangle) \end{aligned} \quad (12)$$

which form a completely entangled basis in \mathbb{H}_1 . One of those states, $|\psi_0\rangle = |0\rangle$, formally corresponds to the EPR state in (5).

Taking into account that the general state (8) of the spin-1 system can be formally represented in the form of the two-qubit state

$$|\psi\rangle = \psi_{\uparrow\uparrow}|\uparrow\uparrow\rangle + \psi_{\downarrow\downarrow}|\downarrow\downarrow\rangle + \psi_{\uparrow\downarrow}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

in the symmetric sector, and that the *concurrence* (measure of entanglement in the case of two qubits [27]) has the form

$$C(\psi) = 2|\det[\psi]| = 2|\psi_{\uparrow\uparrow}\psi_{\downarrow\downarrow} - \psi_{\uparrow\downarrow}\psi_{\downarrow\uparrow}|,$$

we can conclude that the amount of entanglement in the CE basis (12) can be measured by the expression

$$\mathcal{C}(\psi) = 2|\psi_{+1}\psi_{-1} - \psi_0^2/2|, \quad (13)$$

which represents the concurrence in the case of a spin-1 system. It is interesting that the concurrence can also be expressed in terms of the total amount of fluctuation (1) as follows:

$$\mathcal{C}(\psi) = \sqrt{\frac{V_{\text{tot}}(\psi) - V_{\text{min}}}{V_{\text{max}} - V_{\text{min}}}}.$$

In terms of the intrinsic parameter φ introduced by equation (8), the concurrence (13) takes the form

$$\mathcal{C}(\psi) = \cos 2\varphi, \quad \varphi \in [0, \pi/4].$$

Similar analysis can also be done in the case of mixed states of a single spin-1 system.

Concerning physical realizations, let us mention first that the three-dimensional entanglement in orbital angular momentum of photons [21, 22] provides an example, illustrating the above theory. Namely, a single photon in a Laguerre–Gauss beam in the state $|m = 0\rangle$ is entangled by itself. Let us stress that in the usual treatment, entanglement with respect to the orbital angular momentum of a pair of photons [21, 22] is discussed.

Consider now a single electric dipole (E1) photon [23, 24], emitted by an atomic transition between the states $|j = 1, m = 0\rangle$ and $|j' = 0, m' = 0\rangle$. Here j and m denote the angular momentum and its projection, specifying an atomic level. According to selection rules, a photon created by this transition carries total angular momentum $j = 1$ and projection $m = 0$. Thus, in view of our results, it is prepared in CE states.

In view of the above interpretation, we can assume that such a photon may decay into a pair of entangled particles. In other words, the electron–positron pair created by the photodecay of the dipole photon with $m = 0$ should be prepared in the CE EPR state (the last state in (5)) with respect to the spin of charged particles. This may be observed in the presence of a strong electric field, which separates the particles with opposite charge and, unlike the magnetic field, does not influence the spin state. Other photon decay processes such as resonance down-conversion and Raman scattering with creation of the entangled pairs can also be described using the above formalism.

A single biphoton [28, 29] can also be considered as a spin-1 particle. In fact, a photon pair created by spontaneous resonance down-conversion and propagating in the same direction (biphoton) cannot be separated in space and time and therefore should be considered as a single *particle* (spin-1) [29]. In the case of a biphoton, the coefficient of the wavefunction ψ_i in (13) can be interpreted as a component of the biphoton polarization vector [30, 31]. Thus, equation (13) gives concurrence for single spin-1 particle.

Thus, we have shown that the single-particle spin-1 system prepared in the state with spin projection $m = 0$ always

manifests complete entanglement, defined in terms of the maximum total amount of quantum fluctuation. This means that those states are less stable than non-CE states, and that the possible decay of those states leads to creation of EPR pairs. The above consideration shows that the notion of the single-particle entanglement as well as the approach used for its description are quite general. In particular, they can be used for analysis of states of photons, quantum liquids, and elementary particles. Similar analysis can be applied to any physical objects with spin $s \geq 1$ [17].

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