IIE Transactions

ISSN: 0740-817X (Print) 1545-8830 (Online) Journal homepage: http://www.tandfonline.com/loi/uiie20

Tree network 1-median location with interval data: a parameter space-based approach

MUHITTIN HAKAN DEMİR , BARBAROS Ç. TANSEL & GERHARD F. SCHEUENSTUHL

IIE Transactions (2005) **37**, 429–439

ISSN: 0740-817X print / 1545-8830 online
DOI: 10.1080/07408170590918164

Tree network 1-median location with interval data: a parameter space-based approach

MUHITTIN HAKAN DEMİR[1], BARBAROS Ç. TANSEL[1] and GERHARD F. SCHEUENSTUHL[2]

[1] *Department of Industrial Engineering, Bilkent University, Bilkent 06533, Ankara, Turkey*
E-mail: {hdemir,barbaros}@bilkent.edu.tr
[2] *Department of Mathematics, University of Southern California, Los Angeles, CA 90089-1113, USA*

Received 1999 and accepted 2001

We consider a family of 1-median location problems on a tree network where the vertex weights are ranges rather than point values. We define a new framework for making sound decisions under uncertainty which is primarily based on the interplay between the points in the tree and the data that induce the family of problems. An important feature of this framework is that it provides a novel understanding of the problem under uncertainty by collectively handling all possible realizations of the weights. The key element is the notion of a region of a optimality. Based on the regions of optimality, we define three optimality criteria and give low-order polynomial methods to compute the associated solution sets.

1. Introduction

We consider the 1-median location problem on a tree network when the vertex weights are ranges rather than point values. To meaningfully pose the problem, consider first the traditional 1-median problem:

$$\min_{x \in T} \sum_{v_i \in V} w_i d(x, v_i), \tag{1}$$

where $T = (V, E)$ is a tree network with vertex set $V = \{v_1, \ldots v_n\}$, edge set E, and w_i is a non-negative constant specifying the demand (per unit time) at vertex i. A facility is to be located at any point $x \in T$, including vertices and interior points of edges, to minimize the weighted sum of the distances ($d(x, v_i)$ is the length of the path between x and v_i). The problem was initially posed by Hakimi (1964) and any optimal solution to it is termed an *absolute 1-median*. Hakimi (1964) showed that at least one vertex optimally solves Equation (1).

In the problem we consider, the vertex weights w_i are no longer point values. We assume instead that each w_i is an *unknown* number in a prespecified interval $[l_i, u_i]$. We assume that $l_i > 0$ for at least one i and term this the *relative interiority* assumption. Without it, the problem turns into the deterministic problem. We further assume that $l_i > 0$ for at least one i. In addition, we assume that the tree does not contain any vertices of degree one or two whose lower and upper bounds are both zero. If this is not the case, the tree can be preprocessed to eliminate such vertices. The preprocessing does not change the solution sets that will be defined in the following.

The motivation for considering the problem with interval weights is to address a host of questions relating to optimality when demands cannot be predicted with a reasonable degree of precision, but lower and upper bounds can be specified that capture their possible realizations. We assume that each w_i will have some realization in its interval $[l_i, u_i]$, but we do not know *a priori* what this particular value will be at the time of deciding where to locate the facility. It is clear that a location which may be optimal for some realization of the weights may be far from optimality for other realizations. This makes the problem of choosing a location for the facility a non-trivial one.

Traditional ways of dealing with uncertainty can be grouped into three major categories. The first and more widely used way is to utilize expectations. In the expectation approach, the weights are replaced by their expected values, and a deterministic 1-median problem is solved. This may be a reasonable approach if one is interested in the average performance of the system in the long run. In that case, many different realizations of the weights occur with associated probabilities and the expectation approach is justified. However, if there is no historical data on the demands, then it may be extremely difficult to give probability estimates for possible realizations. Even if probability estimates can be made, the expectation-based 1-median location may be severely suboptimal if the realized demands significantly differ from the expected demands. A derivative of this approach is the expectation-variance approach that tries to minimize the expectation plus a multiple of the variance term. This approach requires more detailed information about the probabilities and was basically developed for portfolio analysis in which data on expected values and risks of a small number of discrete alternatives is easier to assess (Markowitz *et al.*, 2000).

A second way to deal with uncertainty is to use a postoptimality approach by first solving a point value problem and then performing a sensitivity analysis. There are two drawbacks associated with this approach. The first one is that it requires the solution of a deterministic problem with some *assumed* data. If the assumed data is the expected demand, then this approach suffers from the same drawbacks as the expectation approach. If the data that is used is not the expected demand, then it is not clear what it should be. The second drawback has to do with the kind of information the postoptimality analysis provides. Typically, a range analysis is performed to determine the range of values of a given w_i for which the found optimal location remains optimal. This assumes that the rest of the data remains constant at their *a priori* fixed values. Hence, the information provided by range analysis is quite limited. If, on the other hand, all weights are allowed to vary simultaneously, then serious computational difficulties may arise. Even if the computational difficulties can be overcome, the postoptimal analysis provides quite limited information in that it focuses on a *single* location and then computes a neighborhood around the *assumed* data within which this location remains optimal.

The third approach, which is probably more in line with the kind of modeling perspectives that we have in mind, is the minimax regret approach. This approach puts emphasis on the worst that can happen when the unknown demands are realized. In this sense, the minimax approach attempts to provide the best protection against the most severe suboptimality that is possible. The minimax regret approach is emerging as a new way of dealing with uncertainty (Kouvelis *et al.*, 1993; Chen and Lin, 1994; Gutierrez and Kouvelis, 1995; Kouvelis and Yu, 1995; Averbakh and Berman, 1996; Variaktarakis and Kouvelis, 1999). One drawback associated with the minimax regret criterion is its overemphasis on the choice of a *single* location that is expected to provide the best protection against the *worst* possible occurrence of the data. In a typical situation, the realized data will be different from the worst possible occurrence. There are many situations in which it is desirable for a decision-maker to be able to choose from among candidate locations that have similar worst-case performances, based on some other criteria. A second drawback has to do with the definition of the regret. The regret can be defined either as the deviation from the optimal value (absolute regret), or as the ratio of the deviation to the optimal value (relative regret). Even though these two regret criteria seem to be closely related, and hence are expected to propose solutions that are not too different from one another, examples can easily be constructed where the two different regret solutions are quite far apart with the solution for the absolute regret criterion performing quite poorly in terms of the relative regret criterion and *vice versa*. A third drawback is an excessive dependence of the minimax regret approach on the edge lengths. Examples can easily be constructed to demonstrate that the absolute regret (the deviation from optimality) can be made arbitrarily bad by an appropriate choice of edge lengths. Example 2 in Section 3 demonstrates some of these points.

Our interest in this paper is to propose a new modeling approach to deal with uncertainty. The approach we propose does not require any probability estimates and can be used for problems with no demand history as long as reasonable lower and upper bounds can be determined for possible demands. Such bounds can be based on expert judgement and educated guesses. Our approach also avoids solving a point value (deterministic) problem. In this sense we do not need the expectations, nor do we need an assumed data for a postoptimality-type analysis. In fact, we do not focus on any particular realization of the data. Instead, we use the weight intervals in an *a priori* sense and identify locations that have a good potential for optimality. If the weight intervals satisfy certain conditions, we identify locations that are optimal in a strong sense. If these conditions are not fulfilled, we identify good candidate locations that collectively perform far better than any single-point solution. We exploit this fact and identify strongly optimal single-point solutions if the actual implementation decision for the facility can be postponed to some extent by allocating funds for preparatory investment in potentially good locations. In a certain sense, this relates to two-stage stochastic optimization with recourse (Birge and Louveaux, 1997) if one views the first stage as the choice of a set of candidate locations and the second stage as the choice of an optimal solution within this set when the demands become known. Despite the apparent similarity, we do not use an expected value approach which is the traditional way of dealing with uncertainty in stochastic optimization. Instead, we try to find a set of locations in the first stage such that they provide maximum coverage against uncertainty. Hence, we deviate from the expectation-based two-stage stochastic optimization in that we seek to find a minimum cardinality set of locations, one of which will optimally respond regardless of the realization of the data. We also deviate from the minimax regret philosophy by shifting our emphasis from a single-point location that optimizes relative to the *worst* possible occurrence of demands to a set of locations that collectively account for *all* possible occurrences of the demands. Hence, our approach is more focused on providing a sound framework that uncovers different aspects of an uncertain situation than in proposing a single-point location that may be "optimal" in a narrow sense.

The paper is organized as follows In Section 2, we define the concept of regions of optimality and the related optimality criteria. In Section 3, we give a comparison of the minimax regret criteria with the proposed criteria. In Section 4, we first give a characterization of the regions of optimality, then give an analysis of weak, permanent, and unionwise permanent solutions. Low-order polynomial algorithms to construct these solution sets are also provided in this section. The paper ends with concluding remarks in Section 5.

2. Regions of optimality and optimality criteria

The key element that interconnects the different solution concepts that we propose in this paper is the concept of a "region of optimality." This concept requires a switch of viewpoint from a *search space* to a *parameter space*. The search space in this problem is the tree network T on which we are searching for a point x to locate the facility. The parameter space, on the other hand, is the space R^n which supplies the data $(w_1 \ldots, w_n)$. We are particularly interested in the subset of R^n consisting of the realizable weight vectors, i.e., the hyperrectangle $H = \{(w_1, \ldots, w_n): l_i \leq w_i \leq u_i$ for all $i = 1, \ldots, n\}$. We refer to H as the *uncertainty set* or the *source set*. Let $P_{\mathbf{w}}$ be the instance of the problem stated in Equation (1) corresponding to the weight vector $\mathbf{w} = (w_1, \ldots, w_n)$. The set of realizable instances of Equation (1) constitutes a family of problem instances $P_H \equiv \{P_{\mathbf{w}}: \mathbf{w} \in H\}$. We now associate a certain subset H_x of the uncertainty set with each point x in the search space T. H_x is defined to be the set of $(w_1, \ldots, w_n) \in H$ such that x optimally solves $P_{\mathbf{w}}$. We refer to H_x as the *region of optimality* of x. The definition implies that x optimally solves Equation (1) for all weight vectors in H_x whereas x is strictly suboptimal for all weight vectors in $H - H_x$.

Example 1. Consider a tree consisting of a single edge $[v_1, v_2]$ with $w_i \in [l_i, u_i]$, $i = 1, 2$. Figure 1 illustrates the uncertainty set H as the rectangular region defined by these bounds. If we draw a 45° line passing through the origin, then the subset of H that lies below or on this line is the region of optimality of v_1. Similarly, H_{v_2} is the portion of H that lies on or above the 45° line. The region of optimality of any interior point x, on the other hand, is the 45° line segment enclosed in H.

This simple example illustrates a number of concepts. Observe first that there is no point in the tree whose region of optimality covers the entire uncertainty set. Hence, no point provides total protection against uncertainty. Another observation we can make is that all interior points of the edge provide a zero area coverage against uncertainty and thus are inferior to vertex locations in terms of the amount of coverage against uncertainty. Of the two vertices, v_1 has a larger region of optimality and thus appears to provide a better protection against uncertainty. Even though this tempts one to locate the facility at v_1, one should be cautious in doing so because the realization of the weights (w_1, w_2) may be outside H_{v_1} in which case suboptimality of v_1 is inevitable. How severe this suboptimality is may play a role in deciding where to locate the facility. If, additionally, a probability distribution on H is given, then it might be more appropriate to examine probability-weighted areas in comparing the relative merits of different locations. This requires integration over subregions of H which may present computational difficulties.

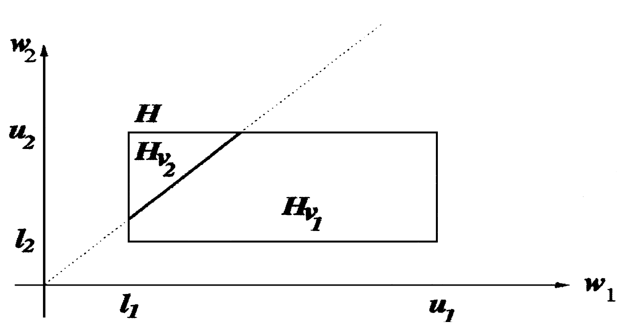

Fig. 1. Illustration of the regions of optimality of a single-edge tree.

A final interesting observation we can make which is not too obvious is the following. Suppose that we currently have enough funds to invest in two pieces of land, one of which will house the actual facility and the other will either be resold or used for some other need that may arise in the future. We assume that the actual implementation of where to build the facility will be made after w_1, w_2 are known. There are two motivating factors why we want to invest now. One is to go through the initial preparatory phase in due time to be able to quickly build the facility later. The second is to make sure that the best location is indeed available at the time at which (w_1, w_2) become known. The decision we face right now is the following: which two locations must we invest in as a working set of locations? The best choice for the two points is the pair of vertices since regardless of which (w_1, w_2) is realized, one vertex will be optimal and will house the actual facility whereas the other one is discarded. Naturally, however, in a problem with $n > 2$ vertices, the choice of two locations as a working set may be considerably more difficult. In that case, we would be looking for two locations whose regions of optimality jointly cover the uncertainty set. If no such two locations exist, it may be necessary to look for three or more locations that collectively cover the uncertainty set.

Based on the region of optimality, we propose the following solution concepts:

1. *Weak solution*: $x \in T$ is a weak solution if and only if $H_x \neq \emptyset$.
2. *Permanent solution*: $x \in T$ is a permanent solution if and only if $H_x = H$.
3. *Unionwise permanent solution:* $U \subset T$ is a unionwise permanent solution if and only if $U_{x \in U} H_x = H$.

Define the *weak set* and the *permanent set* to be the set of weak and permanent solutions, respectively.

As evident from the definitions, all points outside the weak set have no chance of being optimal. A weak solution is optimal for at least one choice of the data whereas a permanent solution is a point which is optimal for all choices of the data. It is clear that weak solutions always exist, and that at least one vertex qualifies as a weak solution as a consequence of the vertex optimality theorem due to Hakimi (1964). Note, however, that no point in the tree may qualify as a permanent solution. For example, if T consists of a single edge $[v_1, v_2]$, with $w_1 \in [1, 2]$ and $w_2 \in [0, 1]$, then vertex v_1 is a permanent solution. If we change the weight interval of vertex v_2 to $[0, 2]$, then there is no permanent solution.

A unionwise permanent solution is a set of points which collectively behave like a permanent solution. Consider a problem where there is no permanent solution. In such a situation, it may so happen that there may be a collection of vertices which supplies an optimal solution regardless of the realization of the data. A crude example of this is the single-edge tree mentioned above where $w_1 \in [1, 2]$ and $w_2 \in [0, 2]$. In this tree, $\{v_1, v_2\}$ is a unionwise permanent solution. Clearly, the existence of a unionwise permanent solution is always guaranteed e.g., the entire vertex set. In fact, another set of vertices which always qualifies as a unionwise permanent solution is the set of vertices that are the weak solutions. The latter set has fewer vertices (in general) than the total number of vertices and is certainly more desirable than the entire vertex set in terms of the suggested number of potential locations for which an initial investment must be made.

There are several reasons for considering unionwise permanent solutions. The main premise for considering unionwise permanent solutions is to identify an initial set of locations one of which is guaranteed to respond optimally whatever course of action might be taken by the external environment. The idea here is to keep this set of locations as a working set of alternatives, then decide, later, which particular location will house the actual facility and which ones will be utilized for secondary or supporting purposes or be discarded. The idea of a working set certainly makes sense in the context of a firm that is making plans for launching a new product line. Initially, it is hard to give point estimates for demands for a new product line whereas it is relatively easy to construct interval demands based on pessimistic and optimistic estimates. Based on the interval demands, a working set of locations for the facility can be identified using the unionwise permanent solution methodology given in this paper. Given a unionwise permanent solution, information on available pieces of land for the active set of vertices can be gathered and negotiations can be carried out to purchase or lease the land and, or the infrastructure for the facility while making arrangements for market surveys and feasibility studies to better assess the demands. Sound market studies may typically take on the order of several months, sometimes a few years, and can narrow down the initial demand intervals into much tighter ranges. At that point, the best location from the working set relative to the narrowed-down intervals or point estimates for demands will be known. The idea is to make sure that this location will be on hand at the time it is decided to go ahead and build the facility. Even though there are several costs associated with keeping a "live" set of locations until the time of deciding the actual location, it is reasonable to make this investment if the associated sunk costs are relatively small in comparison to long-term gains that come from an optimal site selection.

The question arises as to what one must do when the unionwise permanent solution is a large set. There is no easy answer to this question. One possible strategy is to look for ways of refining the data by means of market surveys so that the resulting unionwise permanent solution is small enough to make the additional investment for the live set affordable. If this does not work, then a secondary optimization may be proposed that suffices with partial coverage against uncertainty while staying within a given budget limit. Suppose, for example, a working set of r locations can be kept alive within the available budget, whereas the problem has a unionwise permanent solution consisting of $q > r$ locations. Since the budget permits only r, a reasonable way to do so is to select r of the q locations that provide maximum collective coverage against uncertainty. This can be formulated as a knapsack problem of the form max $\Sigma_{i=1}^{q} c_i x_i$ subject to $\Sigma_{i=1}^{q} x_i \le r$, $x_i \in \{0, 1\}$ where

$$x_i = \begin{cases} 1 & \text{if the } i\text{th element of the unionwise permanent} \\ & \text{solution is selected} \\ 0 & \text{otherwise.} \end{cases}$$

Here, c_i is the volume of the region of optimality of the ith candidate in the working set. This problem has a greedy solution: rank the c_is in non-increasing order and select the first r of them. If different locations in the candidate set require different investment amounts, then the maximum coverage against uncertainty can be found by solving the knapsack problem max $\Sigma_{i=1}^{q} c_i x_i$ subject to $\Sigma_{i=1}^{q} F_i x_i \le b$, $x_i \in \{0, 1\}$ where b is the budget limit and F_i is the investment required for the ith candidate.

The weak, permanent, and unionwise permanent solution concepts and other solution concepts that may be derived from these are based on the relative merits of regions of optimality rather than anything else. For example, it is not difficult to show that the region of optimality of an interior point of an edge of the tree is a subset of the region of optimality of each endpoint of that edge. Consequently, if the size of the region of optimality of a point is taken to be an indicator of how well the point performs in terms of providing protection against uncertainty, then interior points are inferior to endpoints of edges. In this sense, the new optimality criteria suggest a vertex dominance property which is in direct contrast with the minimax regret criterion for which the typical solution is an interior point. Another possibility that receives emphasis due to the use of regions of optimality is the concept of how densely a region of optimality fills the entire uncertainty set. If H_x fills a major portion of H, then x may be considered to be a good solution. The best such solution can be found by defining an auxiliary optimization problem of the form:

$$\max_{x \in T} \frac{\text{Vol}(H(x))}{\text{Vol}(H)},$$

where Vol $(\cdot)$ refers to the volume of a set defined over the lowest dimensional subspace that contains H with a positive volume. A third possibility suggested by the regions of optimality is the possibility of somehow shrinking the uncertainty set H to a smaller subset H' by exercising partial control over the demands (e.g., via promotional efforts, advertisement, pricing) so that the new set H' can be more densely covered by some regions of optimality than H. Doing so may lead to a permanent solution x (i.e., $H_x = H'$) while no such x exists relative to H. For example, in Fig. 1, if an advertisement campaign is launched at v_1 and the lower limit on demand (i.e., the guaranteed sales level) is increased to a new level $l_1' \geq u_2$, then v_1 becomes a permanent solution. Even if H' does not admit a permanent solution, such shrinkage helps to reduce the cardinality of unionwise permanent solutions in computable ways. It seems that the notion of the region of optimality has an inherent drive to propose different and measurable ways of quantifying the relative merits of different locations based on a decomposition of the uncertainty set.

As is evident from the above discussion, the approach proposed in this paper is developed for problems where each realization of demands is as likely as any other. In the absence of any probability information, this is a reasonable assumption. This assumption also implies that a uniform distribution is assumed on the hyperrectangle H. In reality, certain correlations may exist between demands which imply that certain points or regions of the hyperrectangle H have a greater likelihood of occurrence than others. It is easy to see that the nonuniformity assumption on H does not affect the proposed solution concepts. Suppose now, an n-dimensional density function $h(\cdot)$ defined on H is available with $\Pi(S) \equiv \int_S h(w_1 \ldots, w_n)\mathrm{d}w_1 \ldots \mathrm{d}w_n$ supplying the probability that the realized demands are in some subset S of H. If we take S as the region of optimality H_x of some point $x \in T$, then $\Pi(H_x)$ gives the probability that point x optimally responds to nature's choice of demands. With this measure, it is possible now to differentiate between different points in the tree in terms of their probability of optimality. If $h(\cdot)$ is the uniform density, then $\Pi(H_x)$ is simply the volume of H_x; otherwise $\Pi(H_x)$ is the probability-weighted volume of H_x. If, for example, we are interested in the most probable optimal solution, this amounts to solving the auxiliary optimization problem:

$$\max_{x \in T} \Pi(H_x).$$

This problem is in all likelihood computationally demanding and is not considered in this paper. The reason we mention it in passing is that the availability of probability information on demands leads to probability-weighted volumes of regions of optimality which in turn leads to an additional basis of comparison between different points or sets of points of the tree. We reiterate, however, that the weak, permanent, and unionwise permanent sets proposed in this paper are invariant under $h(\cdot)$.

3. Comparison to the minimax regret approach

The minimax regret approach evaluates points in the tree based on the maximum deviation from optimality in terms of the objective function. The minimax regret 1-median problem on a tree was initially studied by Kouvelis *et al.* (1995) and solved in $O(n^3)$ time by Chen and Lin (1994). For a point x in T, define:

$$r(x) \equiv \max_{\mathbf{w} \in H} \left(\sum_{v_i \in V} w_i d(x, v_i) - \min_{v_k \in V} \sum_{v_i \in V} w_i d(v_k, v_i) \right),$$

to be the maximum regret associated with x. The *absolute deviation problem* is posed as follows:

$$\min_{x \in T} r(x).$$

Similarly, the *relative deviation problem* looks for a location that minimizes the maximum relative regret and is posed as follows:

$$\min_{x \in T} r_{\mathrm{D}}(x) \equiv \min_{x \in T} \max_{\mathbf{w} \in H} \left(\sum_{v_i \in V} w_i d(x, v_i) - \min_{v_k \in V} \sum_{v_i \in V} w_i d(v_k, v_i) / \min_{v_k \in V} \sum_{v_i \in V} w_i d(v_k, v_i) \right).$$

Consider the single-edge tree $[v_1, v_2]$ with $w_i \in [l_i, u_i]$, $i = 1$, where $l_1 < u_2$ and $l_2 < u_1$. For any point x in the tree, let δ be the length of the edge segment connecting v_1 and x ($0 \leq \delta \leq L$ where L is the length of the edge). Then

$$\begin{aligned} r(x) &= \max_{l_i \leq w_i \leq u_i, i=1,2} (w_1\delta + w_2(L - \delta) - \min\{w_2 L - w_1 L\}) \\ &= \max\{(u_1 - l_2)\delta, (u_2 - l_1)(L - \delta)\}, \end{aligned}$$

which gives the optimal solution:

$$\delta^* = \frac{(u_2 - l_1)L}{(u_1 - l_1) + (u_2 - l_2)},$$

with

$$r^* = \frac{(u_1 - l_2)(u_2 - l_2)L}{(u_1 - l_1) + (u_2 - l_2)}.$$

In this example, the optimal solution to the absolute deviation problem is an interior point defined by δ^* unless $l_1 \geq u_2$ or $l_2 \geq u_1$ in which case the optimal location is v_1 or v_2. This is in direct contrast with the vertex dominance property suggested by the regions of optimality mentioned earlier in Fig. 1 In general, whereas the minimax regret problem typically proposes interior points as optimal solutions, the region-of-optimality-based approaches dismiss

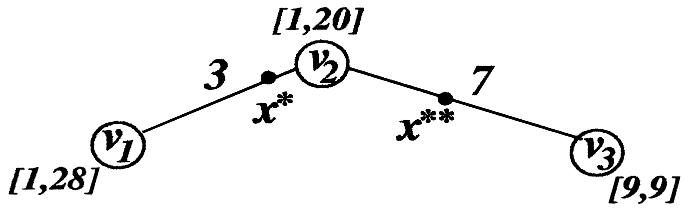

Fig. 2. The tree for example 2.

interior points as inferior points since they have a zero volume coverage of the uncertainty set.

Another key observation we can make from the above example is that both the optimal location δ^* and the optimal regret value r^* are proportional to L. This is again in direct contrast with performance measures based on regions of optimality since the regions of optimality and, consequently, the solutions based on them are invariant with the edge length. This result, which can be easily justified by the edge invariance property of Goldman (1971), is true not only for the single-edge tree under consideration, but also for arbitrary trees. A similar analysis would reveal that region-of-optimality-based solutions are more sensitive to changes in the demand data than minimax regret solutions. Similar conclusions hold for the relative deviation problem.

An inherent characteristic of the minimax regret problem is its strict adherence to proposing a single point as a solution whereas region-of-optimality-based approaches typically propose a number of good locations that collectively perform substantially better than any single-point solution. A concrete example of this is the concept of a unionwise permanent solution, e.g., the solution $\{v_1, v_2\}$ discussed earlier in relation to Fig. 1.

Example 2. Consider the three-vertex tree in Fig. 2. The weight intervals associated with vertices, and the edge lengths are as indicated in the figure. The locations that solve the absolute regret and the relative regret problems are x^* and x^{**}, respectively (shown in the figure). The subedge $[v_1, x^*]$ has length 2.81 and subedge $[v_2, x^{**}]$ has length 3.67. Table 1 gives the absolute regret and relative regret values for these points and the three vertices. The last column in the same table gives the percentage of coverage of the region of optimality of each given point which can be confirmed from Fig. 3. This figure gives H, H_{v_1}, H_{v_2}, H_{v_3}, H_{x^*}, $H_{x^{**}}$ in the (w_1, w_2) plane (since w_3 is a singleton.)

Table 1. Regret values and percentage coverages

Point	*Absolute regret*	*Relative regret*	*Percent coverage*
v_1	84	4.47	31.5
v_2	**54**	2.88	**62.3**
v_3	273	**2.08**	6.2
x^*	**50.7**	3.05	0
x^{**}	143.2	1.37	0

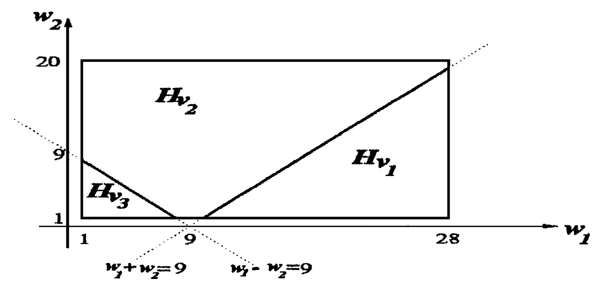

Fig. 3. The regions of optimality for example 2.

Observe in Table 1 that the absolute and the relative regret solutions provide a zero volume coverage against uncertainty (since they are interior points). Table 1 reveals that the three criteria given in the table (absolute regret, relative regret, and percent coverage) are in a fair amount of conflict. For example, the winner of the absolute regret criterion, x^*, performs quite poorly in relative regret and extremely poorly in percent coverage. Likewise, the winner of the relative regret criterion, x^{**}, performs very poorly in absolute regret, and extremely poorly in percent coverage. The winner of the vertex-restricted absolute regret criterion, v_2, is also the winner for the percent coverage criterion. Its absolute regret is very marginally above that of the unrestricted solution (54 versus 50.7) whereas its percent coverage is well above the percent coverage of x^*. The performance of v_2 in terms of the relative regret criterion is also quite good. Hence, v_2 performs very well in two of the criteria and reasonably well in the third. On the other hand, the same type of conclusion does not hold for v_3 which is the optimal vertex-restricted solution for the relative regret criterion. This vertex performs extremely poorly in the remaining two criteria. Another interesting feature revealed by the table is that vertex v_1 is dominated by vertex v_2 in all of the three criteria and hence would be dismissed as an inferior solution if a vector optimization approach were used. However, the elimination of v_1 seems to be an incorrect decision if we are interested in making a preparatory investment in two locations. Despite its mediocre performance in each of the three criteria, v_1, together with v_2, is definitely a good choice in this regard since these two vertices collectively cover almost the entire uncertainty set (93.8%).

4. Analysis

4.1. *Characterization of regions of optimality*

Consider a point x of the tree. If x is a vertex v_k, the deletion of v_k from T together with its incident edges results in as many disjoint components as the degree of v_k. If x is an interior point, then the deletion of x from T together with the two subedges incident to it results in two components. In either case, we refer to each resulting component as a *subtree rooted at* x and denote them as $T_x^1, \ldots, T_x^p$ where p is the number of edges (subedges) incident on x. For any subtree T_x^i, define:

$$w(T_x^i) = \sum_{v_j \in T_x^i} w_j,$$

$$L(T_x^i) = \sum_{v_j \in T_x^i} l_j,$$

and

$$U(T_x^i) = \sum_{v_j \in T_x^i} u_j.$$

To characterize the region of optimality H_x of x, we first consider the deterministic 1-median problem $P_{\mathbf{w}}$ defined by some weight vector $\mathbf{w} = (w_1, \ldots, w_n)$. Goldman (1971) showed that if a subtree S of T contains at least half the total weight of the tree, then it contains an optimal solution for $P_{\mathbf{w}}$ We refer to this as Goldman's majority theorem. It can be shown that this is also a necessary condition for a subtree to contain an optimal solution. This leads to the following characterization of optimality for the deterministic problem.

Lemma 1. *Let $x \in T$ (where x may be an interior point or a vertex location) and $T_x^1, \ldots, T_x^p$ be the subtrees rooted at x then*:

i) *x solves $P_{\mathbf{w}}$ iff $\mathbf{w}(T_x^i) \le \frac{1}{2}\mathbf{w}(T)$ for $\boldsymbol{i} = 1, \ldots, p$ iff $\mathbf{w}(T_x^i) \le \mathbf{w}(T - T_x^i)$ for $i = 1, \ldots, p$;*
ii) *x is the unique optimal solution to $P_{\mathbf{w}}$ iff all inequalities in i hold as strict inequalities.*

Proof. Both parts follow from Goldman's majority theorem. ■

Consider now the problem with interval weights. It follows from Lemma 1 that the region of optimality H_x of a point x in T is the solution set of the following inequality system in the variables $w_1, \ldots, w_n$:

$$\sum_{v_j \in T - T_x^i} w_j - \sum_{v_j \in T_x^i} w_j \ge 0, \qquad i = 1, \ldots, p, \tag{2a}$$

$$l_j \le w_j \le u_j, \qquad j = 1, \ldots, n. \tag{2b}$$

Observe that H_x is the intersection of the cone defined by the first p inequalities and the hyperrectangle H defined by the bounding inequalities.

4.2. *Weak solutions*

Weak solutions are locations that have non-empty regions of optimality. Denote the weak set by $S_{\mathbf{w}}$. Points outside the weak set have no chance of being optimal and can be eliminated from consideration. The weak set may include vertices as well as interior points. The next theorem characterizes the structure of the weak set.

Theorem 1. *The weak set is a subtree.*

Proof. Assume that v_p, v_q are two vertices in the weak set. Then there is a pair of weight vectors, say, $\mathbf{w}^p$ and $\mathbf{w}^q$ in H such that v_p is optimal for the problem defined by $\mathbf{w}^p$ and v_q is optimal for the problem defined by $\mathbf{w}^q$. Let x be a point on the path connecting v_p and v_q. Consider the time parametric problem defined by $\mathbf{w}(t) = \mathbf{w}^p + t(\mathbf{w}^q - \mathbf{w}^p)$, $t \in [0, 1]$. Erkut and Tansel (1992) have shown that there is a time point $t_x \in [0, 1]$ such that x is optimal for the problem defined by $\mathbf{w}(t_x)$. Since $\mathbf{w}^p, \mathbf{w}^q \in H$, $\mathbf{w}(t_x) \in H$. Hence, x is a weak solution. ■

As a consequence of Theorem 1, once leaf vertices of $S_{\mathbf{w}}$ are identified, it is simple to construct $S_{\mathbf{w}}$ as a subtree spanned by its leaf vertices.

The next lemma gives the necessary and sufficient conditions for a subtree to contain a weak solution.

Theorem 2. *Let T_x^i be a subtree rooted at $x \in T$. T_x^i contains a weak solution iff $U(T_x^i) \ge L(T - T_x^i)$.*

Proof. (Necessity.) Assume that T_x^i contains a weak solution, x. x is optimal for some weight vector $\mathbf{w} \in H$. By Lemma 1, $\mathbf{w}(T_x^i) \ge \mathbf{w}(T - T_x^i)$. Since $\mathbf{w} \in H$, $\mathbf{w}(T_x^i) \le U(T_x^i)$ and $\mathbf{w}(T - T_x^i) \ge L(T - T_x^i)$. These inequalities imply that $U(T_x^i) \ge L(T - T_x^i)$.

(Sufficiency.) Assume that $U(T_x^i) \ge L(T - T_x^i)$. Consider the weight vector $\mathbf{w}$ constructed by setting the weights of the vertices in T_x^i at their upper bounds and the weights of the remaining vertices at their lower bounds. Clearly, $\mathbf{w} \in H$. By Lemma 1, T_x^i contains an optimal solution to the problem defined by $\mathbf{w}$. Thus, T_x^i contains a weak solution. ■

The following corollary gives the necessary and sufficient conditions for a leaf vertex to belong to $S_{\mathbf{w}}$.

Corollary 1. *Let v_t be a leaf vertex of T. $v_t \in S_{\mathbf{w}}$ iff $u_t + l_t \ge L(T)$.*

Based on the foregoing results, we construct the weak set using the following tree trimming algorithm which is a generalization of the tree-trimming algorithm of Goldman (1971). In the algorithm, the notation $[v_t, v_p)$ stands for the set of all points on the edge connecting v_t and v_p except v_p.

Algorithm Weak

Step 1. Initial $S_{\mathbf{w}} = T$.
Step 2. Choose some leaf vertex v_t of T and find its unique neighbor v_p.
Step 3. (a) If $u_t + l_t \ge L(T)$, then mark v_t as a leaf vertex of $S_{\mathbf{w}}$.
(b) Otherwise delete $[v_t, v_p)$ from T and update $u_p \leftarrow u_p + u_t$, $l_p \leftarrow l_p + l_t$.
Step 4. If all leaf vertices are marked, stop; otherwise Goto Step 2.

Step 3 of the algorithm applies a test for inclusion into $S_{\mathbf{w}}$ with modified bounds. The test is clearly equivalent to the condition of Theorem 2 since the deleted vertices are

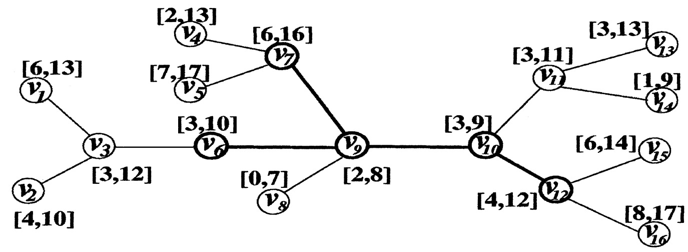

Fig. 4. The tree for example 3.

certainly not in $S_{\mathbf{w}}$. This justifies the correctness of the algorithm. Step 3(a) of the algorithm requires the computation of the sum of the lower bounds of the weights for $T - v_t$ where v_t is a leaf vertex. This can be done *a priori* in $O(n)$ time. Weight updates in Step 3(b) are done in constant time. The algorithm terminates after at most $n - 1$ iterations. Hence, the time bound for the algorithm is $O(n)$.

The following example demonstrates the computation of the weak set.

Example 3. Consider the tree in Fig. 4 with 16 vertices. The lower and upper bounds on the weights of the vertices are as shown in the figure. The vertices are inspected in the order $v_1, v_2, v_3, v_6, v_4, v_5, v_7, v_8, v_{13}, v_{14}, v_{11}, v_{15}, v_{16}, v_{12}$ by the algorithm Weak. v_6, v_7, v_{12} are found to be in $S_{\mathbf{w}}$; hence the weak set is the subtree spanned by these vertices (shown in bold in Fig. 4). The weak set for this example is quite small when compared to the whole tree.

4.3. *Permanent solutions*

A permanent solution is a location that is optimal for every choice of weights within the given lower and upper bounds. If such a solution exists, then its region of optimality covers the uncertainty set H. In this sense, such solutions are strongly optimal solutions, but they may or may not exist. Denote the permanent set by S_{p}.

Theorem 3. *Let x be an interior point of some edge $[v_p, v_q]$. Then $x \notin S_{\mathrm{p}}$.*

Proof. Let x be an interior point and assume $x \in S_{\mathrm{p}}$. Denote the two subtrees rooted at x by T_x^1 and T_x^2. Consider the weight vectors $\mathbf{w}^1, \mathbf{w}^2 \in H$ such that $\mathbf{w}^1$ is constructed by setting all weights at their upper bounds and $\mathbf{w}^2$ is constructed by setting the weights of the vertices in T_x^1 at their upper bounds and the weights of the vertices in T_x^2 at their lower bounds. Since $x \in S_{\mathrm{p}}$, x is optimal for the problems defined by $\mathbf{w}^1$ and $\mathbf{w}^2$. Optimality of x for $\mathbf{w}^1$ implies $U(T_x^1) = U(T_x^2)$, and optimality of x for $\mathbf{w}^2$ implies $U(T_x^1) = L(T_x^2)$. This implies $U(T_x^2) = L(T_x^2)$, that is $u_j = l_j$ for all $v_j \in T_x^2$. Similarly, choosing $\mathbf{w}^3 \in H$ such that weights of the vertices in T_x^1 are set at their lower bounds, and the weights of the vertices in T_x^2 at their upper bounds, optimality of x for $\mathbf{w}^1$ and $\mathbf{w}^3$ implies that $u_j = l_j$ for all $v_j \in T_x^2$. Hence, $u_j = l_j$ for all $v_j \in V$, which contradicts the relative interiority assumption. ■

Corollary 2. *Only vertex locations are candidates for being permanent solutions.*

Corollary 3. *S_{p} is either empty or consists of a single vertex.*

As a consequence of Corollary 3, one can stop the search for the elements of S_{p} as soon as a vertex that belongs to S_{p} is found. The necessary and sufficient conditions for a vertex to belong to S_{p} are given by the next theorem.

Theorem 4. *Let v_k be a vertex in T and T_k^i, $i = 1, \ldots, p$ be the subtrees rooted at v_k. Then, $S_{\mathrm{p}} = \{v_k\}$ if and only if $L(T) \geq U(T_k^i) + L(T_k^i)$ for $i = 1, \ldots, p$.*

Proof. (Necessity.) Let v_k be a permanent solution with p being the number of edges incident at v_k. Consider the p weight vectors $\mathbf{w}^1, \ldots, \mathbf{w}^p$ where $\mathbf{w}^i$ is obtained by setting the weights of the vertices in T_k^i at their upper bounds and weights of the remaining vertices at their lower bounds. Clearly $\mathbf{w}^i \in H$ for each i. Since $v_k \in S_{\mathrm{p}}$, v_k is optimal for $\mathbf{w}^i$, $i = 1, \ldots, p$. That is, $L(T - T_k^i) \geq U(T_k^i)$, $i = 1, \ldots, p$ by Lemma 1. Adding $L(T_k^i)$ to both sides, we have $L(T) \geq U(T_k^i) + L(T_k^i)$ for $i = 1, \ldots, p$.

(Sufficiency.) Assume that $L(T) \geq U(T_k^i) + L(T_k^i)$, $i = 1, \ldots, p$. Subtracting $L(T_k^i)$ from both sides, we have $L(T - T_k)^i \geq U(T_k^i)$, $i = 1, \ldots, p$. Let $\mathbf{w} \in H$. We have that $\mathbf{w}(T - T_k^i) \geq L(T - T_k^i)$ and $\mathbf{w}(T_k^i) \leq U(T_k^i)$ for $i = 1, \ldots, p$. Thus, $\mathbf{w}(T - T_k^i) \geq \mathbf{w}(T_k^i)$, $i = 1, \ldots, p$ which implies that v_k is optimal for the problem defined by $\mathbf{w}$. Since this is true for each $\mathbf{w} \in H$, $v_k \in S_{\mathrm{p}}$. Corollary 3 implies that $S_{\mathrm{p}} = \{v_k\}$. ■

In Theorem 4, if v_k is a leaf vertex, then there is only one subtree, T_k^1, under consideration. Thus, we have:

Corollary 4. *Let v_k be a leaf vertex of T. $S_{\mathrm{p}} = \{v_k\}$ iff $u_k + l_k \geq U(T)$.*

One way of computing the permanent set based on the above results is by applying enumeration on the vertices of T and using the condition in Theorem 4. However, such a procedure has a computational disadvantage. One has to compute total weights (at lower or upper bounds) for each subtree rooted at each vertex of T. We can instead use the following tree trimming procedure that uses the much simpler condition in Corollary 4.

Algorithm Permanent

Step 1. Initial $S_{\mathrm{p}} = \emptyset$.

Step 2. Choose some leaf vertex v_t of T (or of $S_{\mathbf{w}}$) and find its unique neighbor v_p.

Step 3. (a) If $u_t + l_t \geq U(T)$, Goto Step 5.
(b) Otherwise delete $[v_t, v_p)$ from T and update $u_p \leftarrow u_p + u_t$, $l_p \leftarrow l_p + l_t$.

Step 4. If all leaf vertices are tested, stop; otherwise Goto Step 2.

Step 5. Apply the test of Theorem 4 (with the original tree and bounds) to v_t. If the test passes, $S_{\mathrm{p}} = \{v_t\}$, else $S_{\mathrm{p}} = \emptyset$. Stop.

The algorithm stops either with no more leaf vertices remaining to test for or with some leaf vertex in the current tree that passes the test. In the former case, we can conclude that $S_{\mathrm{p}} = \emptyset$. In the latter case, the vertex that passes the test is a permanent solution for the modified (trimmed) tree with the modified bounds, but it *may* or *may not* be a permanent solution for the original tree with the original bounds. Thus, when such a vertex is found, a second test, i.e., the test in Theorem 4 is needed to check whether this vertex is a permanent solution. If the test passes, the vertex under consideration is a permanent solution. Otherwise, $S_{\mathrm{p}} = \emptyset$ since no other remaining vertex in the live tree can pass the test of Corollary 4. It can be seen that the test of Corollary 4 with modified bounds is a relaxation of the test in Theorem 4. Hence, the vertices that do not pass the test in the algorithm cannot be in S_{p}. This justifies the correctness of the algorithm.

Similar to algorithm Weak, the time bound for algorithm Permanent is $O(n)$. If the algorithm identifies a candidate, applying the test in Theorem 4 to this vertex also takes $O(n)$ time. The overall time bound for constructing the permanent set is thus, $O(n)$.

Example 4. Consider the tree in Fig. 5 with 10 vertices. The lower and upper bounds on the weights are as shown in the figure. We first compute $U(T) = \Sigma_{j=1}^{10} u_j = 41$. The vertices are inspected in the order $v_1, v_2, v_3, v_4, v_5, v_7, v_6$ at which point v_6 is found to be the permanent solution for the modified tree via the test of Step 3(a). A check for v_6 using Theorem 4 and the original bounds reveals that v_6 is not a permanent solution for the original tree. Hence, $S_p = \emptyset$.

4.4. *Unionwise permanent solutions*

A unionwise permanent solution is a set of locations one of which is optimal regardless of the realization of the demands. We define a unionwise permanent solution if U is to be a *minimum cardinality* unionwise permanent solution if $|U|U'|$ for all unionwise permanent solutions U'. We define a unionwise permanent solution U to be a *proper* or *minimal* unionwise permanent solution if no proper subset of U qualifies as a unionwise permanent solution. Clearly, a minimum cardinality unionwise permanent solution is a proper unionwise permanent solution, but the converse need not hold. Now, we concentrate on the construction of a unionwise permanent solution.

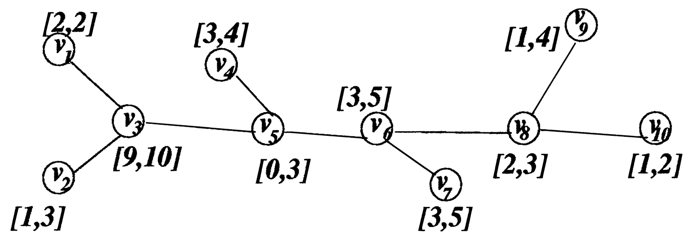

Fig. 5. The tree for example 4.

It is clear from the vertex optimality theorem of Hakimi (1964) that the vertex set of the tree itself is a unionwise permanent solution. Another unionwise permanent solution with a smaller vertex set, in general, is the vertex set of the weak set. Clearly, we prefer proper unionwise permanent solutions to nonproper ones.

Consider a weight vector $\mathbf{w} \in H$. An interior point $x \in T$ is optimal for $\mathbf{w}$ if and only if both of its endpoints are also optimal for $\mathbf{w}$. The "only if" part implies that the region of optimality of an interior point is a subset of the region of optimality of each of its endpoints. Hence, it makes no sense to consider interior points as possible elements of a unionwise permanent solution. Note also that vertices outside the weak set have empty regions of optimality. Therefore, it suffices to consider the vertex elements of the weak set as possible candidates for inclusion in a unionwise permanent solution.

The next theorem states that we may eliminate a leaf vertex of the weak set from consideration for being an element of a proper unionwise permanent solution if the vertex passes the test in Step 3 of algorithm Weak with equality.

Theorem 5. *Let v_k be a leaf vertex of S_{w} and let v_p be the unique adjacent vertex to v_k in S_{w}. Denote by T_k^p the subtree of T rooted at v_k, containing v_p. If $L(T_k^p) = U(T - T_k^p)$, then $H_{v_k} \subset H_{v_p}$.*

Proof. Let v_k and v_p be as given in the theorem. Assume that $L(T_k^p) = U(T - T_k^p)$. Consider the weight vector $\mathbf{w} \in H$ such that $w_i = l_i$ for $v_i \in T_k^p$ and $w_i = u_i$ for $v_i \in T - T_k^p$. Since $L(T_k^p) = U(T - T_k^p)$, $\mathbf{w}(T_k^p) = \mathbf{w}(T - T_k^p)$, hence both v_k and v_p are optimal for $\mathbf{w}$ by Lemma 1. If H_{v_k} is not a subset of H_{v_p}, then there exists a $\mathbf{w}' \in H$ such that v_k is optimal but v_p is not optimal for $\mathbf{w}'$. This implies that $\mathbf{w}'(T_k^p) < \mathbf{w}'(T - T_k^p)$. Since $\mathbf{w}' \in H$, $\mathbf{w}'(T_k^p) \geq L(T_k^p)$ and $\mathbf{w}'(T - T_k^p) \leq U(T - T_k^p)$, we have $L(T_k^p) < U(T - T_k^p)$. This contradicts the assumption that $L(T_k^p) = U(T - T_k^p)$. Hence, $H_{v_k} \subseteq H_{v_p}$. ■

Observe that the condition of Theorem 5 can be repeatedly applied to each leaf vertex of $S_{\mathbf{w}}$ for possible elimination. The resulting set is still a unionwise permanent solution. This follows from the fact that the eliminated vertices have regions of optimality that are covered by regions of optimality of non-eliminated vertices. It can also be shown that the elimination cannot be further repeated for "second generation" leaf vertices that were not leaf vertices of $S_{\mathbf{w}}$ but have become leaf vertices after the elimination of some leaf vertices of $S_{\mathbf{w}}$. This is true since the successive elimination of two adjacent vertices requires that both vertices have zero lower and upper bounds. Such vertices do not exist due to the preprocessing of the initial tree.

Define V' to be the set of vertices of $S_{\mathbf{w}}$ remaining after the elimination of the vertices that fulfill the sufficient condition of Theorem 4. In what follows, we prove that this set is the unique proper (and the unique minimum cardinality) unionwise permanent solution. If V' is a singleton, then the construction of V' implies that it is a (unionwise) permanent solution. Suppose now that V' is not a singleton.

Lemma 2. *Let v_k be a vertex in V' and let $T_k^1, \ldots, T_k^p$ be an enumeration of the subtrees rooted at v_k. Then $U(T - T_k^i) > L(T_k^i)$, $i = 1, \ldots, p$.*

Proof. Pick some arbitrary subtree $T_k^i, i \in \{1, \ldots, p\}$. Let v_t be a leaf vertex of V' that is in $T - T_k^i$ and let v_r be the unique vertex in V' adjacent to v_t. Denote by T_t^r the subtree rooted at v_t, containing v_r. By the construction of V' and by Theorem 5, $U(T - T_t^r > L(T_t^r)$. Since $T_t^r \supseteq T_k^i$, we have $L(T_t^r) \geq L(T_k^i)$ and $U(T - T_t^r) \leq U(T - T_k^i)$. It follows that $U(T - T_k^i) > L(T_k^i)$. ■

Theorem 6. *For each $v_k \in V'$, $\cup_{v_t \in V', t \neq k} H_{vt}$ is a proper subset of H.*

Proof. Let $v_k \in V'$. To prove the claim, it suffices to prove the existence of a weight vector $\mathbf{w}' \in H$ such that v_k uniquely solves $P_{\mathbf{w}'}$. We construct $\mathbf{w}'$ as follows. ■

The fact that $v_k \in S_{\mathbf{w}}$ implies that there exists a $\mathbf{w} \in H$ such that v_k is optimal for $P_{\mathbf{w}}$. If v_k is the unique optimizer for $P_{\mathbf{w}}$ then take $\mathbf{w}' = \mathbf{w}$ and the proof is complete. Otherwise, let $T_k^1, \ldots, T_k^p$ be the subtrees rooted at v_k. Lemma 1 implies that $\mathbf{w}(T - T_k^i) - \mathbf{w}(T_k^i) \geq 0 \, \forall i = 1, \ldots, p$ with at least one equality. With renumbering if necessary, let

$$\mathbf{w}\big(T - T_k^i\big) - \mathbf{w}\big(T_k^i\big) = 0 \quad \text{for} \quad i = 1, \ldots, q, \tag{3}$$

and

$$\mathbf{w}\big(T - T_k^i\big) - \mathbf{w}\big(T_k^i\big) > 0 \quad \text{for} \quad i = q + 1, \ldots, p \quad \text{(if such } i \text{ exists)}, \tag{4}$$

where $1 \leq q \leq p$.

Case 1: $q < p$:

We construct $\mathbf{w}'$ from $\mathbf{w}$ as follows. Let $\epsilon_1 = \min\{\epsilon_2, \epsilon_3\}$ where:

$$\epsilon_2 = \frac{1}{2} \min_{q+1 \leq i \leq p} \big(w\big(T - T_k^i\big) - w\big(T_k^i\big)\big),$$

and

$$\epsilon_3 = \frac{1}{2} \max_{v_j \in \cup_{i=q+1}^p T_k^i} \{u_k - w_k, \max(u_j - w_j)\}.$$

Let $v_{j^*} \in \{v_k\} \cup (\cup_{i=q+1}^P T_k^i)$ be such that $\epsilon_3 = u_j - w_j$. Clearly, $\epsilon_2 > 0$. Either $\epsilon_3 > 0$ or $\epsilon_3 = 0$.

Consider first the case with $\epsilon_3 > 0$. In this case $\epsilon_1 > 0$. Construct $\mathbf{w}'$ by letting $w'_{j^*} = w_{j^*} + \epsilon_1$ and $w'_j = w_j$ for $j \neq j^*$. Observe that $\mathbf{w}(T - T_k^i) - \mathbf{w}(T_k^i) - \mathbf{w}(T_k^i) = \epsilon_1 > 0$ for $i = 1, \ldots, q$ and $\mathbf{w}(T - T_k^i) - \mathbf{w}(T_k^i) \geq \epsilon_1 > 0$ for $i = q + 1, \ldots, p$. Hence, v_k is the unique optimizer for $P_{\mathbf{w}'}$.

Consider now the case with $\epsilon_3 = 0$. In this case, $q > 1$ is not possible. Otherwise, summing the inequalities in Equation (3) for $i = 1, 2$ implies that $w_k = 0$ and $w_j = 0 \, \forall v_j \in \cup_{i=q+1}^P T_k^i$. Since $\epsilon_3 = 0$ implies that $w_k = u_k$ and $w_j = u_j$ for $v_j \in \cup_{i=q+1}^P T_k^i$, we have $u_k = l_k = 0$ and $u_j = l_j = 0$ for $v_j \in \cup_{i=q+1}^p T_k^i$. This is not possible due to the elimination of such vertices in the preprocessing of the initial tree. Hence, $q = 1$. Let $\epsilon_4 = \min\{\epsilon_2, \epsilon_5\}$ where ϵ_2 is as defined above and

$$\epsilon_5 = \max_{v_j \in T_k^1}(w_j - l_j).$$

Let $v_{j^*} \in T_k^1$ be such that $\epsilon_5 = w_{j^*} - l_{j^*}$. We have $\epsilon_5 > 0$ since $\mathbf{w}(T - T_k^1) = \mathbf{w}(T_k^1)$ by assumption and $\mathbf{w}(T - T_k^1) = U(T - T_k^1)$ due to $\epsilon_3 = 0$. Lemma 2 implies that $U(T - T_k^1) - L(T_k^1) > 0$. This together with the last equality, gives $\mathbf{w}(T_k^i) - L(T_k^i) > 0$. Hence, $\epsilon_5 > 0$. We clearly have $\epsilon_2 > 0$. Consequently $\epsilon_4 > 0$. Now construct $\mathbf{w}'$ from $\mathbf{w}$ by letting $w'_{j^*} = w_{j^*} - \epsilon_4$ and $w'_j = w_j$ for $j \neq j^*$. Observe that $\mathbf{w}'(T - T_k^1) - \mathbf{w}'(T_k^1) = \epsilon_4 > 0$ and $\mathbf{w}'(T - T_k^i) - \mathbf{w}'(T_k^i) > \epsilon_4 > 0$ for $i = 2, \ldots, p$. Hence, v_k is the unique optimizer for $P_{\mathbf{w}'}$

Case 2: $q = p$:

If $p = 1$, v_k is a leaf vertex of T hence a leaf vertex of the subtree spanned by V'. Since v_k is not eliminated via Theorem 5, v_k is the unique optimizer for the weight vector $\mathbf{w}'$ where $w'_k = u_k$ and $w'_j = l_j$ for $j \neq k$.

If $p = 2$, summing $\mathbf{w}(T - T_k^1) - \mathbf{w}(T_k^1) = 0$ and $\mathbf{w}(T - T_k^2) - \mathbf{w}(T_k^2) = 0$, we get $w_k = 0$. The case with $u_k = 0$ is not possible. Otherwise, v_k is a degree-two vertex with a zero upper bound and should have been eliminated by the preprocessing of the initial tree. Now, construct the new weight vector $\mathbf{w}'$ by setting $w'_k = u_k$ and $w'_j = w_j$ for $j \neq k$. Since $\mathbf{w}'(T - T_k^i) - \mathbf{w}'(T_k^i) = u_k > 0, i = 1, 2$, v_k is the unique optimizer for $P_{\mathbf{w}'}$.

If $p \geq 3$, then summing $\mathbf{w}(T - T_k^1) - \mathbf{w}(T_k^1) = 0, i = 1, \ldots, p$, we get $w_j = 0$ for all $v_j \in V$. However, this is not possible since $l_j > 0$ for at least one $v_j \in V$.

Hence, we can construct a $\mathbf{w}' \in H$ such that v_k is the unique optimizer for $P_{\mathbf{w}'}$. The conclusion follows. ■

Based on the above results, we have:

Corollary 5. *V' is the unique proper unionwise permanent solution.*

Since a minimum cardinality unionwise permanent solution is also a proper unionwise permanent solution, we have:

Corollary 6. *V' is the unique minimum cardinality unionwise permanent solution.*

Example 5. Consider the tree in Fig. 4. The vertices of the weak set, as found in example 3, are v_6, v_7, v_9, v_{10}, and v_{12}. Checking the iterations of algorithm Weak, we see that the vertices v_6, v_7, and v_{12} pass the test for inclusion into $S_{\mathbf{w}}$ with equality $(u_t + l_t = L(T))$. Hence, eliminating v_6, v_7, and v_{12}, $V' = \{v_9, v_{10}\}$ is the unique proper and minimum cardinality unionwise permanent solution. This example demonstrates that, although the tree under consideration has 16 vertices with rather crude weight estimates, the two vertices v_9 and v_{10} suffice to collectively optimize the locational decision.

5. Conclusions

Most facility location decisions require a long-term commitment to the same location. If the facility under consideration does not have a sufficient demand history, then the lack of point estimates for demands may lead to severely suboptimal decisions that may threaten the long-term existence of the facility. This paper proposes a framework for sound decisions when there is a substantial degree of imprecision associated with demands. The weak solutions proposed in the paper serve to identify locations that deserve further analysis in the location decision-making process. Points outside the weak set are dismissed as inferior locations. The second solution concept is the so-called permanent solution which truly optimizes the system performance if certain conditions in the demand data prevail. If the data fails to satisfy these conditions, then no permanent solution exists and one must look for alternate ways of approaching the problem. To this end, we propose and exploit the notion of unionwise permanent solutions. Such solutions are not single-point solutions but rather constitute an active set of locations that collectively optimize the system performance in a well-defined sense. Modeling perspectives for these solution concepts are given in the paper and exact methods are given for efficiently computing them. Comparisons and contrasts to prevailing traditional methods for dealing with uncertainty have also been discussed.

The framework and solution concepts proposed in this paper find natural extensions in the context of general networks. The problem on a general network may require additional analysis tools to deal with complications that arise from the presence of network cycles. For the case of a single facility, it appears possible to overcome some of the additional difficulties by breaking down the edges of the network into segments defined by edge bottleneck points, thereby taking advantage of certain (treelike) properties associated with these segments. For the case of multiple facilities, the problem is likely to be much more involved than the single facility case both on tree networks and on general networks.

References

Averbakh, I. and Berman, O. (1996) Minmax regret 1-center location on a network with demand uncertainty. Research report, Division of Management and Economics, University of Toronto, Scarborough, Ontario. Canada, M1C 1AA.

Birge, J.B. and Louveaux, F. (1997) *Introduction to Stochastic Programming*, Springer Verlag, New York, NY.

Chen, B. and Lin, C.S. (1994) Robust one median location problem. Research report, Department of Management and Systems, Washington State University, Pullman, WA 99164-4726. USA.

Erkut E. and Tansel B.C. (1992) On parametric medians of trees. *Transportation Science*, **26**(2), 149–156.

Goldman, O.J. (1971) Optimal center locations in simple networks. *Transportation Science*, **5**, 212–221.

Gutierrez, G.J. and Kouvelis, P. (1995) A robustness approach to international sourcing. *Annals of Operations Research*, **59**, 165–193.

Hakimi, S.L. (1964) Optimal locations of switching centers and the absolute centers and medians of a graph. *Operations Research*, **12**, 450–459.

Kouvelis P., Vairaktarakis, G. and Yu, G. (1993) Robust 1-median location on a tree in the presence of demand and transportation cost uncertainty. Research report, College of Business Administration. Marquette University. Milwaukee WI 53233, USA.

Kouvelis P. and Yu, G. (1996) Robust discrete optimization and its applications, Kluwer, Dordrecht. The Netherlands.

Markowitz, H.M., Todd, G.P. and Todd, P. (2000) *Mean-variance in portfolio choice and capital markets*, McGraw Hill.

Vairaktarakis G. and Kouvelis, P. (1999) Incorporating dynamic aspects and uncertainty in 1-median location problems. *Naval Research Logistics*, **46**, 147–168.

Biographies

Muhittin Hakan Demir obtained his M.Sc. and Ph.D. degrees from the Department of Industrial Engineering, Bilkent University, Ankara, Turkey in 1994 and 2001, respectively. He is currently working in a private company in Turkey. His research interests lie in uncetain optimization, facility location, and discrete optimization.

Barbaros C. Tansel is a Professor and the Chairman of the Department of Industrial Engineering at Bilkent University, Ankara, Turkey. He earned his Ph.D. degree in 1979 from the University of Florida (Department of Industrial and Systems Engineering). Prior to his appointment at Bilkent University, he was a visiting faculty member at Georgia Institute of Technology and a faculty member at the University of Southern California. His primary research interests are in location theory, combinatorial optimization, and optimization with imprecise data. He has published in various journals including *Operations Research, Management Science, Transportation Science, IIE Transactions, European Journal of Operational Research, Journal of the Operational Research Society, International Journal of Production Research*, and *Journal of Manufacturing Systems*.