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Godel-type metrics in various dimensions:¨ II. Inclusion of a dilaton field

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Abstract
This is the continuation of an earlier work where Godel-type metrics were¨ defined and used for producing new solutions in various dimensions. Here, a simplifying technical assumption is relaxed which, among other things, basically amounts to introducing a dilaton field to the models considered. It is explicitly shown that the conformally transformed Godel-type metrics can¨ be used in solving a rather general class of Einstein–Maxwell-dilaton-3-form field theories in $D \geqslant 6$ dimensions. All field equations can be reduced to a simple 'Maxwell equation' in the relevant (D − 1)-dimensional Riemannian background due to a neat construction that relates the matter fields. These tools are then used in obtaining exact solutions to the bosonic parts of various supergravity theories. It is shown that there is a wide range of suitable backgrounds that can be used in producing solutions. For the specific case of (D−1)-dimensional trivially flat Riemannian backgrounds, the *D*-dimensional generalizations of the well-known Majumdar–Papapetrou metrics of general relativity arise naturally.

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1. Introduction

Let *M* be a *D*-dimensional manifold with a metric of the form

$$g_{\mu\nu} = h_{\mu\nu} - u_\mu u_\nu. \tag{1}$$

(We take Greek indices to run from 0, 1, ... to D − 1 and our conventions are similar to the conventions of Hawking–Ellis [1].) Here, $h_{\mu\nu}$ is a degenerate D×D matrix with rank equal to D−1. We assume that the degeneracy of $h_{\mu\nu}$ is caused by taking $h_{k\mu} = 0$, where x^k is a fixed coordinate with $0 \leqslant k \leqslant D-1$ (note that x^k does not necessarily have to be spatial), and that

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the rest of $h_{\mu\nu}$, i.e. $\mu \neq k$ or $\nu \neq k$, is dependent on all the coordinates x^α except x^k so that $\partial_k h_{\mu\nu} = 0$. Hence, in the most general case, 'the background' $h_{\mu\nu}$ can effectively be thought of as the metric of a (D − 1)-dimensional non-flat spacetime. As for u^μ, we assume that it is a timelike unit vector, $u_\mu u^\mu = -1$, and that u_μ is independent of the fixed special coordinate x^k, i.e. $\partial_k u_\mu = 0$. The assumptions so far imply that the determinant of $g_{\mu\nu}$ is $g = -u^2_k h$, where *h* is the determinant of the (D − 1) × (D − 1) submatrix obtained by deleting the *k*th row and the *k*th column of $h_{\mu\nu}$, and moreover $u^\mu = -\frac{1}{u_k}\delta^\mu_k$.

The question we ask now is the following: let us start with a metric of the form (1) and calculate its Einstein tensor. Can the Einstein tensor be interpreted as describing the energy–momentum tensor of a physically acceptable source? Does one need further assumptions on $h_{\mu\nu}$ and/or u_μ so that 'the left-hand side' of $G_{\mu\nu} \sim T_{\mu\nu}$ can be thought of as giving an acceptable 'right-hand side', i.e. corresponding to a physically reasonable matter source? As you will see in the subsequent sections, the answer is 'yes' provided that one further demands $h_{\mu\nu}$ to be the metric of an Einstein space of a (D −1)-dimensional Riemannian geometry. We call such a metric $g_{\mu\nu}$ a *Godel-type metric*¨.

Inour firstpaper onthissubject [2], wehave examined thisquestion indetail forthesimple case u_k = constant. For the choice of constant u_k, one finds that u^μ is a Killing vector, thanks to the assumptions stated so far. In [2], we showed that in all dimensions the Einstein equations are classically equivalent to the field equations of general relativity with a charged dust source provided that a simple (D − 1)-dimensional Euclidean source-free Maxwell's equation is satisfied. Then the energy density of the dust fluid is proportional to the Maxwell invariant F^2. We further demonstrated that the geodesics of the Godel-type metrics are described by¨ solutions of the (D −1)-dimensional Euclidean Lorentz force equation for a charged particle. We also discussed the possible existence of examples of spacetimes containing closed timelike and closed null curves which violate causality and examples of spacetimes without any closed timelike or closed null curves where causality is preserved. We showed that the Godel-type¨ metrics with constant u_k provide exact solutions to various kinds of supergravity theories in five, six, eight, ten and eleven dimensions. All these exact solutions are fundamentally based on the vector field u_μ which satisfies the (D − 1)-dimensional Maxwell's equation in the background of some (D − 1)-dimensional Riemannian geometry with metric $h_{\mu\nu}$. It was in this respect that [2] gave not only a specific solution but in fact provided a whole class of exact solutions to each of the aforementioned theories. Moreover in [2], explicit examples of exact solutions were separately constructed for the cases of both trivially flat and non-flat backgrounds $h_{\mu\nu}$, the latter of which included a conformally flat space, an Einstein space and a Riemannian Tangherlini solution in D = 4.

All of the solutions presented in [2] were mathematically simple and some of them were known beforehand; however, [2] certainly did provide a useful, nice and unified treatment of Godel-type solutions. In the present work, we follow on our promise to further consider the¨ case when $u_k \neq$ constant.

We show that the information inherent in the Einstein equations calculated from the conformally transformed Godel-type metric (or the G¨ odel-type metric in a string frame) with¨ a non-constant u_k can be thought of as classically equivalent to that of an Einstein–Maxwelldilaton-3-form theory for dimensions $D \geqslant 6$. The Maxwell and 3-form fields are related to the vector field u_μ in a simple manner, whereas the dilaton field is simply given by $\varphi \equiv \ln|u_k|$. As a consequence of the special construction employed in the formation of the 3-form field and the dilaton, the field equations satisfied by these matter fields are shown to follow from the 'Maxwell equation' that the vector field u_μ satisfies. We also comment on the possible choices of the (D−1)-dimensional Riemannian spacetime backgrounds and derive the

‘Maxwell’ and the ‘dilaton’ equations explicitly in the background $h_{\mu\nu}$ using the ‘Maxwell equation’ for

u_μ. We next show that the Godel-type metrics provide exact solutions to the bosonic parts of¨ various kinds of supergravity theories in six, eight and ten dimensions. In this respect, rather than giving specific solutions, we provide a whole class of exact solutions to each of these supergravity theories depending on the choice of their respective backgrounds. As particular examples, we explicitly construct solutions found by taking the (D − 1)-dimensional flat Riemannian and Riemannian Tangherlini geometries as backgrounds. In the former case, the solutions turn out to be the *D*-dimensional generalizations of the well-known Majumdar–Papapetrou metrics of D = 4 dimensions. Since the D = 3 case is special, we examine it separately and find a rich family of exact solutions then. Some of the long calculations and their technical details are presented in the two appendices following our conclusions.

2. Einstein–Maxwell-dilaton-3-form theories

It readily follows from the assumptions stated in the first paragraph of the introduction that $h_{\mu\nu}u^\nu = 0$ and the inverse of the metric can be calculated as $g^{\mu\nu} = \bar{h}^{\mu\nu} + (-1 + \bar{h}^{\alpha\beta}u_\alpha u_\beta)u^\mu u^\nu + u^\mu(\bar{h}^{\nu\alpha}u_\alpha) + u^\nu(\bar{h}^{\mu\alpha}u_\alpha)$. (2)

Here, $\bar{h}^{\mu\nu}$ is the (D − 1)-dimensional inverse of $h_{\mu\nu}$, i.e. $\bar{h}^{\mu\nu}h_{\nu\alpha} = \bar{\delta}^\mu_\alpha$ with $\bar{\delta}^\mu_\alpha$ denoting the (D − 1)-dimensional Kronecker delta: $\delta^\mu_\alpha = \bar{\delta}^\mu_\alpha + \delta^\mu_k\delta^k_\alpha$.

By defining the Christoffel symbols of $h_{\mu\nu}$ as

$$\gamma^\mu{}_{\alpha\beta} = \tfrac{1}{2}\bar{h}^{\mu\nu}[h_{\nu\alpha,\beta} + h_{\nu\beta,\alpha} - h_{\alpha\beta,\nu}], \tag{3}$$

one finds that the Christoffel symbols of $g_{\mu\nu}$ are given by

$$\Gamma^\mu{}_{\alpha\beta} = \gamma^\mu{}_{\alpha\beta} + \tfrac{1}{2}\left(u_\alpha f^\mu{}_\beta + u_\beta f^\mu{}_\alpha\right) - \tfrac{1}{2}u^\mu(u_{\alpha|\beta} + u_{\beta|\alpha}). \tag{4}$$

Here, we use a vertical stroke to denote a covariant derivative with respect to $\gamma^\mu{}_{\alpha\beta}$ and $f_{\alpha\beta} \equiv \partial_\alpha u_\beta - \partial_\beta u_\alpha$. We further assume that the indices of u_μ and $f_{\alpha\beta}$ are raised and lowered by the metric $g_{\mu\nu}$; thus, e.g., $f^\mu{}_\nu = g^{\mu\alpha}f_{\alpha\nu}$. Note that the definition of $f_{\alpha\beta}$ is left invariant if the ordinary derivatives are replaced by the covariant derivatives with respect to $\gamma^\mu{}_{\alpha\beta}$ (or $\Gamma^\mu{}_{\alpha\beta}$ for that matter). To remove any further ambiguity, we will denote a covariant derivative with respect to $\Gamma^\mu{}_{\alpha\beta}$ by ∇_μ in what follows.

Now one can show that u^μ is not a Killing vector, unlike the constant u_k case examined in [2]. This follows from a number of identities that involve the vector u^μ and its covariant derivative $\nabla_\beta u_\alpha$ that are presented in detail in appendix A.

Let us now define $\phi \equiv \ln|u_k|$ and denote the energy–momentum tensor of this scalar ϕ as

$$T^\phi_{\mu\nu} = (\nabla_\mu\phi)(\nabla_\nu\phi) - \tfrac{1}{2}g_{\mu\nu}(\nabla\phi)^2,$$

where $(\nabla\phi)^2 \equiv g^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi)$. Similarly, one can also define the energy–momentum tensor for the ‘Maxwell field’ $f_{\mu\nu}$ as

$$T^f_{\mu\nu} \equiv f_{\mu\alpha}f_\nu{}^\alpha - \tfrac{1}{4}g_{\mu\nu}f^2,$$

where $f^2 \equiv f^{\alpha\beta}f_{\alpha\beta}$. One can now calculate the Einstein tensor to be

$$\begin{aligned}G_{\mu\nu} = {} & \hat{r}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\hat{r} - \tfrac{1}{2}T^\phi_{\mu\nu} + \tfrac{1}{2}T^f_{\mu\nu} - \nabla_\mu\nabla_\nu\phi + \tfrac{1}{2}g_{\mu\nu}\Box\phi - \tfrac{1}{4}f^2(u_\mu u_\nu + g_{\mu\nu}) \\ & + \tfrac{1}{2}u_\mu\,\mathrm{e}^\phi\nabla_\alpha\left(\mathrm{e}^{-\phi}f^\alpha{}_\nu\right) + \tfrac{1}{2}u_\nu\mathrm{e}^\phi\nabla_\alpha\left(\mathrm{e}^{-\phi}f^\alpha{}_\mu\right) - \tfrac{1}{2}g_{\mu\nu}(u_\beta\,\mathrm{e}^\phi\nabla_\alpha(\mathrm{e}^{-\phi}f^{\alpha\beta})),\end{aligned} \tag{5}$$

where $\varphi \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi$. Here, $\hat{r}_{\mu\nu}$ and $\hat{r}$ denote the Ricci tensor and the Ricci scalar of $\gamma^{\mu}{}_{\alpha\beta}$, respectively. (See appendix A for the details of how (5) is obtained.)

One can already catch glimpses of the field equations for the Einstein–Maxwell-dilaton theory at this stage. However, the $\nabla_\mu\nabla_\nu\varphi$ term is obviously not very pleasing and needs to be discarded if one is to go for such an identification. This can be achieved by utilizing a conformal transformation on the metric $g_{\mu\nu}$. Let $\tilde{g}_{\mu\nu} = e^{2\psi}g_{\mu\nu}$ (hence $\tilde{g}^{\mu\nu} = e^{-2\psi}g^{\mu\nu}$), where ψ is a smooth function. Using the results of [3], one can calculate the Einstein tensor $\tilde{G}_{\mu\nu}$ associated with the metric $\tilde{g}_{\mu\nu}$ and its derivative operator $\tilde{\nabla}_\mu$ in terms of the quantities associated with $g_{\mu\nu}$ to be

$$\tilde{G}_{\mu\nu} = G_{\mu\nu} - (D-2)\nabla_\mu\nabla_\nu\psi + (D-2)\psi_\mu\psi_\nu + (D-2)g_{\mu\nu}\left(\Box\psi + \tfrac{1}{2}(D-3)(\nabla\psi)^2\right).$$

Hence choosing the function ψ as $\psi = \varphi/(2 - D)$, one can eliminate the $\nabla_\mu\nabla_\nu\varphi$ term and, after a few simplifications, obtain

$$\begin{aligned}\tilde{G}_{\mu\nu} &= \frac{1}{2}u_\mu\, e^{\phi}\nabla_\alpha\left(e^{-\phi} f^{\alpha}{}_{\nu}\right) + \frac{1}{2}u_\nu\, e^{\phi}\nabla_\alpha\left(e^{-\phi} f^{\alpha}{}_{\mu}\right) - \frac{1}{2}g_{\mu\nu}(u_\beta\, e^{\phi}\nabla_\alpha(e^{-\phi} f^{\alpha\beta}))\\ &\quad + \hat{r}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\hat{r} + \frac{1}{2}T^{f}_{\mu\nu} - \frac{1}{4}f^2(u_\mu u_\nu + g_{\mu\nu})\\ &\quad - \frac{1}{2}g_{\mu\nu}\Box\phi + \frac{3D-8}{4(D-2)}g_{\mu\nu}(\nabla\phi)^2 + \frac{4-D}{2(D-2)}\phi_\mu\phi_\nu.\end{aligned} \tag{6}$$

Now one has to transform the right-hand side of (6) in such a way that all quantities and especially covariant derivatives are only related to the conformally transformed metric (or the string metric) $\tilde{g}_{\mu\nu}$. The details of this calculation are not very illuminating and so we present them in appendix B. The result is the following:

$$\begin{aligned}\tilde{G}_{\mu\nu} &= \hat{r}_{\mu\nu} + \frac{4-D}{2(D-2)}\tilde{T}^{\phi}_{\mu\nu} + \frac{1}{2}e^{\frac{2\phi}{2-D}}\tilde{T}^{f}_{\mu\nu}\\ &\quad -\frac{1}{2}\tilde{g}_{\mu\nu}\left(e^{-\frac{2\phi}{2-D}}\hat{r} + \tilde{\Box}\phi + \frac{1}{2}e^{\frac{2\phi}{2-D}}\tilde{f}^2 + \tilde{u}_\beta\tilde{\nabla}_\alpha\left(e^{\frac{2\phi}{2-D}}\tilde{f}^{\alpha\beta}\right)\right)\\ &\quad + \frac{1}{2}\tilde{u}_\mu\left(\tilde{\nabla}_\alpha\left(e^{\frac{2\phi}{2-D}}\tilde{f}^{\alpha}{}_{\nu}\right) - \frac{1}{4}e^{\frac{4\phi}{2-D}}\tilde{f}^2\tilde{u}_\nu\right) + \frac{1}{2}\tilde{u}_\nu\left(\tilde{\nabla}_\alpha\left(e^{\frac{2\phi}{2-D}}\tilde{f}^{\alpha}{}_{\mu}\right) - \frac{1}{4}e^{\frac{4\phi}{2-D}}\tilde{f}^2\tilde{u}_\mu\right).\end{aligned} \tag{7}$$

on these quantities are raised using $\tilde{g}^{\mu\nu}$ (see appendix B for details).

Let us now define a 3-form field $\tilde{H}_{\mu\tau\sigma}$ as

$$\tilde{H}_{\mu\tau\sigma} \equiv \tilde{f}_{\mu\tau}\tilde{u}_\sigma + \tilde{f}_{\tau\sigma}\tilde{u}_\mu + \tilde{f}_{\sigma\mu}\tilde{u}_\tau. \tag{8}$$

If we denote the energy–momentum tensor for this 3-form field $\tilde{H}_{\mu\tau\sigma}$ as

We would like to emphasize in passing that here we take $\tilde{u}_\mu = u_\mu, \tilde{f}_{\mu\nu} = f_{\mu\nu}$ and the indices

$$\tilde{T}^{H}_{\mu\nu} \equiv \tilde{H}_{\mu\tau\sigma}\tilde{H}_{\nu}{}^{\tau\sigma} - \tfrac{1}{6}\tilde{g}_{\mu\nu}\tilde{H}^2,$$

where $\tilde{H}^2 \equiv \tilde{H}_{\mu\tau\sigma}\tilde{H}^{\mu\tau\sigma}$, one finds with the help of

$$\begin{aligned}\tilde{H}_{\mu\tau\sigma}\tilde{H}_{\nu}{}^{\tau\sigma} &= -2\,e^{-\frac{2\phi}{2-D}}\tilde{f}_{\mu\tau}\tilde{f}_{\nu}{}^{\tau} + \tilde{f}^2\tilde{u}_\mu\tilde{u}_\nu - 2\,e^{-\frac{4\phi}{2-D}}(\tilde{\nabla}_\mu\phi)(\tilde{\nabla}_\nu\phi)\\ &\quad -2\,e^{-\frac{2\phi}{2-D}}(\tilde{\nabla}_\tau\phi)(\tilde{f}_{\nu}{}^{\tau}\tilde{u}_\mu + \tilde{f}_{\mu}{}^{\tau}\tilde{u}_\nu),\end{aligned} \tag{9}$$

$$\tilde{H}^2 = -3\,e^{-\frac{2\phi}{2-D}}\tilde{f}^2 - 6\,e^{-\frac{4\phi}{2-D}}(\tilde{\nabla}\phi)^2 \tag{10}$$

that (7) can be written as

$$\tilde{G}_{\mu\nu} = \hat{r}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\,e^{-\frac{2\phi}{2-D}}\hat{r} + \frac{1}{4}e^{\frac{4\phi}{2-D}}\tilde{T}^{H}_{\mu\nu} + e^{\frac{2\phi}{2-D}}\tilde{T}^{f}_{\mu\nu} + \frac{1}{D-2}\tilde{T}^{\phi}_{\mu\nu}, \tag{11}$$

provided that

$$\tilde\nabla_\mu\left(e^{\frac{2\phi}{2-D}}\tilde f^{\mu\nu}\right) = \tfrac{1}{2}\,e^{\frac{4\phi}{2-D}}\tilde H^{\nu\tau\sigma}\tilde f_{\tau\sigma}, \tag{12}$$

$$\tilde\Box\phi + \tfrac{1}{2}\,e^{\frac{2\phi}{2-D}}\tilde f^2 + \tfrac{1}{6}\,e^{\frac{4\phi}{2-D}}\tilde H^2 = 0, \tag{13}$$

$$\tilde\nabla_\mu\left(e^{\frac{4\phi}{2-D}}\tilde H^{\mu\nu\alpha}\right) = 0. \tag{14}$$

Before making an interpretation of these equations, let us in fact observe that both (13) and (14) follow from (12). This is seen by noting that (12) can be equivalently written as

$$\tilde\nabla_\mu\tilde f^\mu{}_\nu = \frac{1}{2}\,e^{\frac{2\phi}{2-D}}\tilde f^2\,\tilde u_\nu - \frac{D}{2-D}(\tilde\nabla_\sigma\phi)\tilde f^\sigma{}_\nu. \tag{15}$$

Contracting this equation by $\tilde u^\nu$ and using the identities

$$\tilde u_\mu\,\tilde u^\mu = -e^{-\frac{2\phi}{2-D}} \qquad \text{and} \qquad \tilde u^\mu\,\tilde f_{\mu\nu} = e^{-\frac{2\phi}{2-D}}\tilde\nabla_\nu\phi,$$

one obtains

$$\tilde\Box\phi = (\tilde\nabla\phi)^2 \tag{16}$$

which is equivalent to (13) by (10) itself. $\tilde H^{\nu\tau\sigma}\tilde f_{\tau\sigma} = \tilde f^2\,\tilde u^\nu - 2(\tilde\nabla_\sigma\phi)\,e^{-\frac{2\phi}{2-D}}\tilde f^{\nu\sigma}$, Similarly, using $-u_k\,e^{\frac{2\phi}{2-D}}\tilde u^\alpha$ in (12), contracting this new equation by $\delta^\alpha{}_k$ (which is equal to), adding the copies of the resulting equation that are obtained by taking the cyclic permutations of the three indices, one finds (14).

In fact using $K^\sigma{}_{\mu\nu}$ introduced in appendix B, (12) (or equivalently (15)) can be transformed into

$$\nabla_\mu f^\mu_\nu = \tfrac{1}{2}f^2 u_\nu + 2\phi_\sigma f^\sigma{}_\nu$$

and by using (2) and (4), this can be further written as

$$(\bar h^{\mu\alpha}f_{\alpha\nu})_{|\mu} + (\bar h^{\mu\sigma}u_\sigma\phi_\nu)_{|\mu} = \bar h^{\mu\alpha}f_{\alpha\nu}\phi_\mu + \bar h^{\mu\sigma}\phi_\mu u_\sigma\phi_\nu + \bar h^{\mu\sigma}\phi_\sigma u_{\nu|\mu} \tag{17}$$

in the background $h_{\mu\nu}$. Similarly, (13) (or equivalently (16)) is simply given by

$$\bar\Box_h\,e^{-\phi} \equiv \frac{1}{\sqrt{|h|}}\partial_\mu\left(\sqrt{|h|}\,\bar h^{\mu\nu}\partial_\nu\,e^{-\phi}\right) = 0 \tag{18}$$

in terms of the background $h_{\mu\nu}$. The information inherent in (18) is naturally contained in (17), of course.

In retrospect we have shown that the conformally transformed metric (or the string metric)

$$\tilde g_{\mu\nu} = e^{\frac{2\phi}{2-D}}(h_{\mu\nu} - u_\mu u_\nu),$$

our choice of u_μ (and hence $f_{\mu\nu}$) and our construction of the 3-form field $\tilde H_{\mu\tau\sigma}$ solve the Einstein–Maxwell-dilaton-3-form field equations (11)–(14) in *D* dimensions provided the background $h_{\mu\nu}$ is chosen suitably so that the contribution of the first two terms on the righthand side of (11) can be controlled and given a physically reasonable matter interpretation and the Maxwell equation (17) (and hence the dilaton equation (18)) in the background $h_{\mu\nu}$ are satisfied.

We should perhaps emphasize that the very forms of (12)–(14) all follow naturally from the way the Maxwell field and the 3-form field are constructed; the coefficients that show up in these equations are not in any way determined by a further requirement or a framework such as supersymmetry.

Now let us recall that by assumption u^{μ} is taken as a timelike vector and for the string metric $\tilde{g}_{\mu\nu}$ to have the correct Minkowskian signature, the background $h_{\mu\nu}$ has to necessarily be taken as the metric of a (D − 1)-dimensional Riemannian spacetime. However, note that a crucial requirement here is to be able to choose the background $h_{\mu\nu}$ so that the first two terms on the right-hand side of (11) are kept under control. This requirement indeed constrains the choice of allowable $h_{\mu\nu}$ further. A few of the acceptable backgrounds that first come to mind are as follows: flat Riemannian spacetimes, flat Riemannian Tangherlini solutions, flat Riemannian Myers–Perry solutions, Ricci-flat Kahler manifolds (provided that¨ D −1 is even, of course) and suitably chosen gravitational instanton solutions. Depending on whether one is willing to accommodate an extra perfect fluid source on the right-hand side of (11), one can also try out the de Sitter versions of some of the aforementioned spacetimes.

We should also make a few remarks regarding the similarity between our Godel-type¨ metrics and the metrics used in Kaluza–Klein reductions in string theory. In a typical Kaluza–Klein reduction from *D* dimensions to (D −1) dimensions, the process is usually carried on a spatial dimension and the theory obtained in the lower dimension has the usual Minkowskian signature just like its parent theory. Here, in contrast to what is done in the Kaluza–Klein mechanism, the Godel-type metrics are used in obtaining a¨ *D*-dimensional theory starting from a (D − 1)-dimensional one, and this is done by the use of backgrounds $h_{\mu\nu}$ with Euclidean signature. If anything, such backgrounds could only emerge after a Kaluza–Klein reduction performed on the time direction of a usual Minkowskian signature theory, and hence as objects living in Euclidean signature gravity (or perhaps supergravity) theories. (In this respect, [4] is a pioneering work that studies how Euclidean signature supergravities arise by compactifying D = 11 supergravity or type IIB supergravity on a torus that includes the time direction.)

3. Solutions in various dimensions

Let us now look for simple theories whose field equations contain the Einstein–Maxwelldilaton-3-form field equations (11)–(14) that we have derived in the previous section. The presence of a 3-form itself suggests that one should turn to dimensions $D \geqslant 6$. As will become apparent in what follows, one can use any one of the backgrounds listed in the second last paragraph of section 2 to construct solutions to the bosonic parts of some supergravity theories in six, eight and ten dimensions. We do not claim that these are the only possible theories for which the techniques we present are applicable, the theories we consider here are merely examples. We would also like to stress out that our aim here is just to show how one can use the construction presented in the previous section to obtain solutions to these theories. A detailed analysis of these solutions and a consideration of how much supersymmetry, if any, they preserve is far beyond the scope of this work and, apart from mentioning possible directions in which to look, here we completely refrain from delving into such delicate issues.

3.1. Six dimensions

In our conventions, the bosonic part of the gauged D = 6,N = 2 supergravity [5] reduces to the following field equations when all the scalars of the hypermatter φ^a and the 2-form field $B_{\mu\nu}$ in the theory are set to zero and the dilaton ϕ is taken to be non-constant[1]:

[1] The constant factor in front of the very last term of (4.17) of [5] is incorrect. Here, we fix this and use the correct one.

$$R_{\mu\nu} = 2\,\mathrm{e}^{\sqrt{2}\varphi} F_{\mu\rho} F_{\nu}{}^{\rho} + \mathrm{e}^{2\sqrt{2}\varphi} G_{\mu\rho\sigma} G_{\nu}{}^{\rho\sigma} + 2(\nabla_\mu \varphi)(\nabla_\nu \varphi) - \frac{1}{\sqrt{2}} g_{\mu\nu} \Box \varphi, \quad (19)$$

$$\nabla_\mu \left(\mathrm{e}^{\sqrt{2}\varphi} F^{\mu\nu}\right) = \mathrm{e}^{2\sqrt{2}\varphi} G^{\nu\rho\sigma} F_{\rho\sigma}, \quad (20)$$

$$\nabla_\mu \left(\mathrm{e}^{2\sqrt{2}\varphi} G^{\mu\nu\rho}\right) = 0, \quad (21)$$

$$\phi = \frac{1}{3\sqrt{2}} \mathrm{e}^{2\sqrt{2}\phi} G_{\mu\nu\rho} G^{\mu\nu\rho} + \frac{1}{2\sqrt{2}} \mathrm{e}^{\sqrt{2}\phi} F_{\mu\nu} F^{\mu\nu}. \quad (22)$$

Here all Greek indices run from 0 to 5 and $G_{\mu\nu\rho}$ is given by

$$G_{\mu\nu\rho} = F_{\mu\nu}A_\rho + F_{\nu\rho}A_\mu + F_{\rho\mu}A_\nu \tag{23}$$

and instead of a Yang–Mills field, we have taken an ordinary vector field A_μ to be present. In fact using (22) in (19), one finds that the Einstein tensor satisfies

$$G_{\mu\nu} = 2\,e^{\sqrt{2}\varphi}T^F_{\mu\nu} + e^{2\sqrt{2}\varphi}T^G_{\mu\nu} + 2T^\varphi_{\mu\nu} \tag{24}$$

for this theory. Note the striking resemblance of this theory to the model we described in section 2.

Now let $h_{\mu\nu}$ be any Ricci-flat Riemannian metric in five dimensions so that $\hat{r}_{\mu\nu} = \hat{r} = 0$ and

$$g_{\mu\nu} = 2e^{\sqrt{2}\bar{\phi}}(h_{\mu\nu} - u_\mu u_\nu). \tag{25}$$

Moreover, take $A_\mu = u_\mu$ so that $F_{\mu\nu} = f_{\mu\nu}$ and $G_{\mu\nu\rho} = \tilde{H}_{\mu\nu\rho}$ as in section 2. Then, a careful comparison of (24), (20), (21), (22) with (11), (12), (14) and (13), respectively, shows that these two sets of equations are identical provided $\varphi = -2(\sqrt{2}\bar{\phi} + \ln 2)$.

Hence, the conformally transformed Gödel-type metric (25) becomes an exact solution of $D = 6, N = 2$ supergravity theory provided $h_{\mu\nu}$ is chosen as any Ricci-flat Riemannian metric in five dimensions and u_μ satisfies (17) in this background. As stated earlier, our aim here is just to show how the techniques of section 2 can be used to find solutions. A comparison of the solutions that can be obtained here with the (supersymmetric) ones given in [6, 7] (and the references therein) and an investigation of how much supersymmetry they preserve is certainly worth further study.

3.2. Eight dimensions

The bosonic part of the gauged $D = 8, N = 1$ supergravity theory coupled to *n* vector multiplets [8] has field equations which are very similar to the field equations of the gauged $D = 6$, $N = 2$ supergravity theory that we have examined in subsection 3.1. Taking an ordinary vector field instead of a Yang–Mills field and setting the 2-form field B_{MN} equal to 0, one similarly has

$$G_{MNP} = F_{MN}A_P + F_{NP}A_M + F_{PM}A_N, \tag{26}$$

where now capital Latin indices run from 0 to 7. We also set all the scalars in the theory to zero apart from the dilaton σ which we take as non-constant. These assumptions lead to the following field equations (see (26) of [8]):

$$R_{MN} = 2\,e^{\sigma} F_{MP} F_N{}^{P} + e^{2\sigma} G_{MPS} G_N{}^{PS} + \tfrac{3}{2}(\nabla_M\sigma)(\nabla_N\sigma) - \tfrac{1}{2} g_{MN}\,\Box\sigma, \tag{27}$$

$$\nabla_M(e^{\sigma} F^{MN}) = e^{2\sigma} G^{NPS} F_{PS}, \tag{28}$$

$$\nabla_M(e^{2\sigma} G^{MNP}) = 0, \tag{29}$$

$$\sigma = \tfrac{2}{9}\,e^{2\sigma} G_{MNP} G^{MNP} + \tfrac{1}{3}\,e^{\sigma} F_{MN} F^{MN}, \tag{30}$$

in our conventions. In fact using (30) in (27), one finds that the Einstein tensor satisfies

$$G_{MN} = 2\,e^{\sigma} T^F_{MN} + e^{2\sigma} T^G_{MN} + \tfrac{3}{2} T^{\sigma}_{MN}. \tag{31}$$

0,rˆ = 0 and

$$g_{MN} = 2e^{\sigma}(h_{MN} - u_M u_N). \tag{32}$$

Now let h_{MN} be any Ricci-flat Riemannian metric in seven dimensions so that $\hat{r}_{MN}$ = Moreover, take $A_M = \tilde{u}_M$ so that $F_{MN} = \tilde{f}_{MN}$ and $G_{MNP} = \tilde{H}_{MNP}$ similarly to what we did in subsection 3.1. Then, a careful comparison of (31), (28), (29), (30) with (11), (12), (14) and (13), respectively, shows that these two sets of equations are identical provided $\varphi = -3(\sigma + \ln 2)$.

One again arrives at the conclusion that the conformally transformed Godel-type metric¨ (32) yields an exact solution to the gauged $D = 8, N = 1$ supergravity with matter couplings provided h_{MN} is chosen as any Ricci-flat Riemannian metric in seven dimensions and u_M satisfies (17) in this background. The conditions on u_M under which these solutions are supersymmetric should be examined further.

3.3. Ten dimensions

In our conventions, the following field equations belong to type IIB supergravity with only a graviton, a dilaton and a 3-form gauge field present [9]:

$$R_{MN} = \tfrac{1}{2}(\nabla_M\sigma)(\nabla_N\sigma) + \tfrac{1}{4}\,e^{\sigma} F_{MPQ} F_N{}^{PQ} - \tfrac{1}{48}\,e^{\sigma} g_{MN} F_{PQR} F^{PQR}, \tag{33}$$

$$\nabla_M(e^{\sigma} F^{MNP}) = 0, \tag{34}$$

$$\sigma = \tfrac{1}{12}\,e^{\sigma} F_{MNP} F^{MNP}. \tag{35}$$

Here, all Latin indices run from 0 to 9. Using (33) and (35), the Einstein tensor can be shown to satisfy

$$G_{MN} = \tfrac{1}{4}\,e^{\sigma} T^F_{MN} + \tfrac{1}{2} T^{\sigma}_{MN}. \tag{36}$$

Comparing (36), (34), (35) with (11), (14) and (13) (with the 2-form field $\tilde{f}$ set identically equal to 0 in the latter set), respectively, these two sets of equations can be shown to be identical provided $\tilde{H} = F$ and $\varphi = -2\sigma$.

Therefore, provided h_{MN} is chosen as any Ricci-flat Riemannian metric in nine dimensions and u_M satisfies (17) in this background, a 2-form field f_{MN} can be formed in the usual way which in turn can be used in constructing a 3-form field

$$F_{MNP} = f_{MN}u_P + f_{NP}u_M + f_{PM}u_N$$

that can be used together with the conformally transformed Godel-type metric¨

$$g_{MN} = e^{\sigma/2}(h_{MN} - u_M u_N)$$

to solve (33)–(35).

4. Special solutions constructed by simple choices of backgrounds

Just to give an idea as to how one goes about finding actual solutions, we will now consider as simple examples the solutions that can be constructed by taking the (D − 1)-dimensional flat Riemannian and Riemannian Tangherlini geometries as the backgrounds. These can, of course, be considered as solutions of the supergravity theories examined in section 3.

4.1. Spacetimes with (D − 1)*-dimensional flat Riemannian solutions as backgrounds*

Obviously, the simplest possible background one can think of is the (D − 1)-dimensional flat Riemannian geometry. Now without loss of generality, let us assume that the special fixed coordinate has been chosen so that k = 0. Thus our background reads

$$ds^2_{D-1} = (dx^1)^2 + (dx^2)^2 + \cdots + (dx^{D-1})^2 = \bar{\delta}_{ij}\, dx^i\, dx^j,$$

where, as in a standard spacetime decomposition, Latin indices i,j range from 1 to D − 1. While we are at it, let us furthermore assume that u_μ has only one non-zero component and so $u_\mu = e^{\varphi}\delta_\mu{}^0$, where φ depends on all x^α except for x^0. It immediately follows that the only non-vanishing components of $f_{\mu\nu}$ are given by $f_{i0} = -f_{0i} = \varphi_i\, e^{\varphi}$.

Now it is very easy to see that (18) yields $\partial^2 e^{-\varphi} = 0$, where $\partial^2 \equiv \bar{\delta}^{ij}\partial_i\partial_j$ is the usual Laplacian operator of Euclidean geometry. One can now also show that (17) is identically satisfied. Thus, in this simplest example, the *D*-dimensional line element is found as

$$ds^2 = e^{\frac{2\phi}{2-D}}[(dx^1)^2 + \cdots + (dx^{D-1})^2 - e^{2\phi}(dx^0)^2], \qquad (37)$$

where $e^{-\varphi}$ is any harmonic function that solves $\partial^2 e^{-\varphi} = 0$.

Actually the Godel-type metric (¨ 37) is the *D*-dimensional generalization of the wellknown Majumdar–Papapetrou [10, 11] metric of D = 4 dimensions, and has recently been used in constructing multi shell models for removing the multi black hole singularities of such spacetimes [12]. We should emphasize that the Godel-type metric (¨ 37) is a solution to the supergravity models considered in section 3. Specifically, we find the following:

(i) D = 6 model described in subsection 3.1.

Equation (37) now reduces to

$$ds^2 = e^{-\varphi/2}[(dx^1)^2 + \cdots + (dx^5)^2] - e^{3\varphi/2}(dx^0)^2.$$

As carefully described in subsection 3.1, in this case the field content is properly covered by taking $A_\mu = u_\mu = e^{\varphi}\delta_\mu^0$ and hence $F_{\mu\nu} = f_{\mu\nu}$ and $G_{\mu\nu\rho} = \tilde{H}_{\mu\nu\rho}$ as in section 2. The harmonic function $e^{-\varphi}$ can be taken as

$$e^{-\phi} = 1 + \sum_{k=1}^{N} \frac{m_k}{|\vec{r} - \vec{a}_k|^3}$$

in analogy with the Majumdar–Papapetrou construction and this can be used to find the dilaton field ϕ by the relation $\varphi = -2(\sqrt{2}\bar{\phi} + \ln 2)$ (as described in subsection 3.1). Here, the parameters m_k can be thought of as denoting the masses of the N point particles located at $\vec{r} = \vec{a}_k$ (see [12] for details).

(ii) D = 8 model described in subsection 3.2.

A similar analysis as done for D = 6 above can also be carried out for D = 8 by following the discussion in subsection 3.2. One finds that this time (37) reduces to

$$ds^2 = e^{-\varphi/3}[(dx^1)^2 + \cdots + (dx^7)^2] - e^{5\varphi/3}(dx^0)^2,$$

and taking $A_M = \tilde{u}_M = e^{\varphi}\delta_\mu^0$ (and hence $F_{MN} = \tilde{f}_{MN}$ and $G_{MNP} = \tilde{H}_{MNP}$), the model described in subsection 3.2 is solved provided the harmonic function $e^{-\varphi}$ is now taken as

$$e^{-\phi} = 1 + \sum_{k=1}^{N} \frac{m_k}{|\vec{r} - \vec{a}_k|^5}$$

with a similar analogy and notation as before. The dilaton field σ is found by using the relation $\varphi = -3(\sigma + \ln 2)$ in this case.

(iii) D = 10 model described in subsection 3.3.

Finally, the D = 10 theory described in subsection 3.3 can be covered in a similar manner. The metric

$$ds^2 = e^{-\varphi/4}[(dx^1)^2 + \cdots + (dx^9)^2] - e^{7\varphi/4}(dx^0)^2,$$

with the properly constructed 3-form field $F_{MNP} = \tilde{H}_{MNP}$ (as explained in subsection 3.3) and the harmonic function $e^{-\varphi}$

$$e^{-\phi} = 1 + \sum_{k=1}^{N} \frac{m_k}{|\vec{r} - \vec{a}_k|^7},$$

defined in analogy with the previous cases, make up the field content of the D = 10 model of subsection 3.3. The dilaton field σ in this case is simply given by $\varphi = -2\sigma$.

Note that the ansatz we used for u_μ was rather simple. One can, of course, use more complicated ones such as $u_\mu = u_i\delta^i_\mu + e^\phi\delta^0_\mu$, with the functions u_i and φ depending on any number of x^j but x^0, which will cause some form of rotation in the spacetime described by the metric $\tilde{g}_{\mu\nu}$. It is clear that there is a vast number of possibilities that can be tried out.

4.2. *Spacetimes with* (D − 1)*-dimensional Riemannian Tangherlini solutions as backgrounds*

Consider the line element corresponding to the (D − 1)-dimensional Riemannian Tangherlini solution

$$ds^2_{D-1} = \zeta(r)dt^2 + \frac{dr^2}{\zeta(r)} + r^2\, d\Omega^2_{D-3}, \tag{38}$$

where

$$\zeta(r) = 1 - 2mr^{4-D}, \qquad (D \geqslant 4),$$

m > 0 is the constant mass parameter, $d\Omega^2_{D-3}$ is the metric on the (D − 3)-dimensional unit sphere and *r* is the usual radial coordinate that defines this sphere [2, 13]. (Even though the background is flat when D = 4, we keep it for the discussion that will follow.)

Let the special fixed coordinate x^k be x^{D-1} and assume that $u_\mu = u(r)\delta^0_\mu + e^{\phi(r)}\delta^{D-1}_\mu$. Then $f_{\mu\nu} = (\delta^r_\mu\delta^0_\nu - \delta^0_\mu\delta^r_\nu)u' + (\delta^r_\mu\delta^{D-1}_\nu - \delta^{D-1}_\mu\delta^r_\nu)\phi'\, e^\phi$, where a prime denotes a derivative with respect to *r*. Now (18) implies that

$$(r^{D-3}\zeta(e^{-\phi})')' = 0, \tag{39}$$

which is easily integrated as

$$e^{-\phi} = a + \begin{cases} \dfrac{b}{2m(D-4)}\ln\zeta, & (D \geqslant 5) \\ \dfrac{b}{1-2m}\ln r, & (D = 4) \end{cases}, \tag{40}$$

for some real integration constants *a* and *b*. One can now calculate the Christoffel symbols $\gamma^\mu{}_{\alpha\beta}$ of the background and use these in (17) to see that the only nontrivial component of (17) (independent of the information provided by (39)) is obtained when ν = 0 and in that case u(r) satisfies

$$u'' + \left(\frac{D-3}{r} - 2\phi'\right)u' + \left(\frac{\phi'\zeta'}{2\zeta}\right)u = 0, \qquad (D \geqslant 5), \tag{41}$$

$$u'' + \left(\frac{1}{r} - 2\phi'\right)u' = 0, \qquad (D = 4). \tag{42}$$

Unfortunately, for ζ(r) and φ(r) given as above, one cannot find an explicit solution to the second-order linear ordinary differential equation (41) in terms of known functions. However, one can implicitly show that a physically acceptable solution to (41) exists and a numerical solution can be

$$u(r) = c_1 - \frac{c_2}{b}(1-2m)\, e^{\phi(r)} = c_1 - \frac{c_2(1-2m)^2}{b(a(1-2m) + b\ln r)}, \qquad (D = 4$$

constructed by starting at $r \to \infty$ and coming in toward the outermost singularity at some $r_s > 0$. Fortunately, things are a little better for $D = 4$ since (42) can be integrated to give),

for some real integration constants c_1 and c_2.

The D-dimensional line element reads

$$
\mathrm{d}s^2 = \begin{cases} \mathrm{e}^{\frac{2\phi}{2-D}} \left(\zeta(r)\,\mathrm{d}t^2 + \dfrac{\mathrm{d}r^2}{\zeta(r)} + r^2\,\mathrm{d}\Omega^2_{D-3} - (u(r)\,\mathrm{d}t + \mathrm{e}^{\phi(r)}\,\mathrm{d}x^{D-1})^2 \right), & (D \geqslant 5), \\ \dfrac{a(1-2m)+b\ln r}{1-2m} \left((1-2m)\,\mathrm{d}t^2 + \dfrac{\mathrm{d}r^2}{1-2m} + r^2\,\mathrm{d}\theta^2 - \left(\left(c_1 - \dfrac{c_2(1-2m)^2}{b(a(1-2m)+b\ln r)}\right)\mathrm{d}t + \dfrac{1-2m}{a(1-2m)+b\ln r}\,\mathrm{d}x^3 \right)^2 \right), & (D = 4). \end{cases} \tag{43}
$$

It is instructive to look at the two invariants $\tilde{R}$ and $\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu}$ at this point to see the singularity structure of the spacetimes described by (43). In the simpler $D = 4$ case, one finds that

$$\tilde{R} = \frac{b^2(1-2m)\left(1-2m+c_1^2\right)}{2r^2(a(1-2m)+b\ln r)^3}$$

and

$$\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} = \frac{b^4(1-2m)^2\left[3(1-2m)^2 + 3c_1^4 - 4(1-2m)c_1^2\right]}{4r^4(a(1-2m)+b\ln r)^6}.$$

Depending on how the integration constants a, b, c_1 and the mass parameter m are related to each other, one finds singularities at $r = 0$ and at $r = r_s$ for which $\phi(r_s) \to \infty$, i.e., at $r_s = \exp(-a(1-2m)/b) > 0$.

Similarly, even though there exists no explicit solution $u(r)$ of (41), one would expect to find singularities at $r = 0$, at the zeros of the metric function $\zeta(r)$ and at the r values where $\phi(r) \to \infty$ when $D \geqslant 5$.

It naturally follows that the conformally transformed Gödel-type metric (43) can obviously be used as a solution to the supergravity theories that we have described in subsections 3.1–3.3 by fixing the dimension D accordingly. However, one still needs to understand the physical relevance of the geometries described by (43) in the context of these theories.

5. $D = 3$ solutions with two-dimensional backgrounds

Let us now examine how things go for the special $D = 3$ case separately. When $D = 3$, introduction of a 3-form field leads to a triviality since any totally antisymmetric 3-tensor then has to be proportional to the Levi-Civita tensor density, . Hence, we go back to the Einstein tensor (7) of the conformally transformed metric $\tilde{g}_{\mu\nu}$ and find that it simplifies to give

$$\tilde{G}_{\mu\nu} = \tfrac{1}{2}\hat{r}\tilde{u}_\mu\tilde{u}_\nu + \tfrac{1}{2}\tilde{T}^\phi_{\mu\nu} + \tfrac{1}{2}\,\mathrm{e}^{-2\phi}\tilde{T}^f_{\mu\nu} - \tfrac{1}{2}\tilde{g}_{\mu\nu}\left(\tilde{\Box}\phi + \tfrac{1}{2}\,\mathrm{e}^{-2\phi}\tilde{f}^2 + \tilde{u}_\beta\tilde{\nabla}_\alpha(\mathrm{e}^{-2\phi}\tilde{f}^{\alpha\beta})\right)$$
$$+ \tfrac{1}{2}\tilde{u}_\mu\left(\tilde{\nabla}_\alpha\left(\mathrm{e}^{-2\phi}\tilde{f}^\alpha{}_\nu\right) - \tfrac{1}{4}\,\mathrm{e}^{-4\phi}\tilde{f}^2\tilde{u}_\nu\right) + \tfrac{1}{2}\tilde{u}_\nu\left(\tilde{\nabla}_\alpha\left(\mathrm{e}^{-2\phi}\tilde{f}^\alpha{}_\mu\right) - \tfrac{1}{4}\,\mathrm{e}^{-4\phi}\tilde{f}^2\tilde{u}_\mu\right), \tag{44}$$

where we have used the fact that the Einstein tensor of a two-dimensional metric vanishes identically.

At this stage there is no *a priori* way of choosing the Maxwell equation. For simplicity though, let us assume that it is

$$\tilde{\nabla}_\mu\left(\mathrm{e}^{-2\phi}\tilde{f}^\mu{}_\nu\right) = a\,\mathrm{e}^{-4\phi}\tilde{f}^2\tilde{u}_\nu, \tag{45}$$

where a is an arbitrary function. In a manner similar to how we showed that (13) follows from (12), one can show that (45) implies

$$\tilde{\Box}\phi = \left(a - \tfrac{1}{2}\right)\mathrm{e}^{-2\phi}\tilde{f}^2. \tag{46}$$

When (45) and (46) are used in (44), one gets

$$\tilde{G}_{\mu\nu} = \left(\tfrac{1}{2}\hat{r} + \left(a - \tfrac{1}{4}\right)\mathrm{e}^{-4\phi}\tilde{f}^2\right)\tilde{u}_\mu\tilde{u}_\nu + \tfrac{1}{2}\,\tilde{T}^\phi_{\mu\nu} + \tfrac{1}{2}\,\mathrm{e}^{-2\phi}\tilde{T}^f_{\mu\nu}. \tag{47}$$

Togetherwith(45)(and(46)), (47)canbeinterpretedasdescribinganEinstein–Maxwell-scalar field theory coupled with a charged dust fluid in three dimensions with energy density

$$\rho = \tfrac{1}{2}\,\hat{r} + \left(a - \tfrac{1}{4}\right)\mathrm{e}^{-4\phi}\tilde{f}^2.$$

Now let us take the special fixed coordinate x^k as $x^0 \equiv t$ without any loss of generality. It is well known that any two-dimensional metric is conformally flat and can be put in the form $h_{ij} = e^{2\sigma(x,y)}\bar{\delta}_{ij}$. (Here i,j = 1,2, we take $x^1 \equiv x, x^2 \equiv y$ and in what follows we use $\partial^2\sigma \equiv \bar{\delta}^{ij}\partial_i\partial_j\sigma$ and $(\partial\sigma)^2 \equiv \bar{\delta}^{ij}(\partial_i\sigma)(\partial_j\sigma)$.) Then, the Ricci scalar of h_{ij} is simply $\hat{r} = -2e^{-2\sigma}\partial^2\sigma$. Let us furthermore assume that

$$u_\mu = e_{\varphi(x,y)}\delta_{\mu 0}.$$

Put altogether, these assumptions lead to the conformally transformed metric

$$ds^2 = -dt^2 + e^{2(\sigma-\varphi)}(dx^2 + dy^2), \tag{48}$$ and $f_{\mu\nu} = \left(\delta^i_\mu\delta^0_\nu - \delta^0_\mu\delta^i_\nu\right)(\partial_i\phi)\,\mathrm{e}^\phi$. Now a careful calculation gives $\tilde{f}^2 = -2(\partial\varphi)^2\, e^{4\varphi-2\sigma}$.

This in turn implies that $\rho = \left(\left(\tfrac{1}{2} - 2a\right)(\partial\phi)^2 - \partial^2\sigma\right)\mathrm{e}^{-2\sigma}$. Using these, one finds that (45) (or equivalently (46)) is satisfied provided

$$\partial^2\varphi = (1 - 2a)(\partial\varphi)^2. \tag{49}$$

For constant a, this means that $\partial^2(e^{(2a-1)\varphi}) = 0$ when $a \neq 1/2$ and $\partial^2\phi = 0$ when a = 1/2.

Therefore, any function $\varphi(x,y)$ for which $e^{(2a-1)\varphi}$ is harmonic in the (x,y) variables solves (49) when a is a constant and $a \neq 1/2$. When a = 1/2, φ itself must be a harmonic function.

So given the value of a and ρ, one can first find a suitable φ and using this determine the function σ(x,y) to construct the metric (48) that exactly solves the theory described by equations (47), (45) and (46).

As a specific example, suppose that there is no dust fluid present, i.e. the energy density ρ has been set to zero. Now using (49) in the expression for ρ, one finds that

$$\partial^2\sigma = \frac{1-4a}{2(1-2a)}\partial^2\phi \qquad \text{for} \qquad a \neq \frac{1}{2}, \tag{50}$$

and given $a \neq 1/2$, one can first determine φ through (49) and then use this in (50) to determine σ.

6. Conclusions

We have used the previously introduced Godel-type metrics to find solutions to the Einstein ̈ field equations coupled with a Maxwell-dilaton-3-form field theory in $D \geqslant 6$ dimensions. By construction the matter fields were related to the vector field u_μ and their field equations were shown to follow from the 'Maxwell equation' for u_μ. We showed that there is a vast number of possibilities for the choice of the (D − 1)-dimensional Riemannian spacetime backgrounds $h_{\mu\nu}$ and that one can find exact solutions to the bosonic field equations of supergravity theories in six, eight and ten dimensions by effectively reducing these equations to a single 'Maxwell equation' (17) in the relevant background $h_{\mu\nu}$. Specifically, when $h_{\mu\nu}$ is the (D − 1)-dimensional flat Riemannian geometry, the solutions found happen to be the *D*-dimensional generalizations of the D = 4 dimensional Majumdar–Papapetrou metrics. The D = 3 case was also shown to admit a family of solutions describing a Maxwell-dilaton field.

It would be worth studying to seek other theories for which the techniques we have employed here are applicable. A detailed analysis of the solutions we have found and an investigation of how much supersymmetry, if any, they preserve is one possible future direction to look at. It would also be very interesting to try out more general (D − 1)dimensional Riemannian spacetimes (some of which we have articulated in the second to last paragraph of section 2) as backgrounds, to solve (17) using these and to find possibly new solutions. Another attractive avenue to look at would be to try out Godel-type metrics for ̈ which $\partial_k g_{\mu\nu} \neq 0$, in contrast to what has been assumed so far.

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Appendix A. Preliminaries for obtaining (5)

In this appendix, we present some calculations which are useful in the derivation of (5). By definition

$$\nabla_\beta u_\alpha = \partial_\beta u_\alpha - \Gamma^\mu{}_{\alpha\beta} u_\mu = u_{\alpha|\beta} - C^\mu{}_{\alpha\beta} u_\mu,$$

where we have introduced

$$C^{\mu}{}_{\alpha\beta} \equiv \Gamma^{\mu}{}_{\alpha\beta} - \gamma^{\mu}{}_{\alpha\beta} = \tfrac{1}{2}\left(u_{\alpha} f^{\mu}{}_{\beta} + u_{\beta} f^{\mu}{}_{\alpha}\right) - \tfrac{1}{2} u^{\mu}(u_{\alpha|\beta} + u_{\beta|\alpha}).$$

Bydefining$\varphi \equiv \ln|u_k|$anddenoting$\varphi_\mu = \nabla_\mu \varphi$, onecanshowthatthefollowingidentities hold:

$$u^\alpha \nabla_\beta u_\alpha = 0, \quad (A.1) \quad u^\mu f_{\mu\nu} = \varphi_\nu, \quad (A.2)$$

$$u^\mu \varphi_\mu = 0. \tag{A.3}$$

The first one is found by using $u_\mu u^\mu = -1$ whereas the next two follow from the form of u^μ, the assumption that u_μ is independent of x^k and the definition of φ. Similarly, one obtains the following formulae regarding the derivatives of u_μ:

$$\nabla_\alpha u_\beta = \tfrac{1}{2}[f_{\alpha\beta} - u_\alpha \phi_\beta - u_\beta \phi_\alpha] \quad ,$$

$$(A.4) \quad \nabla_\alpha u^\alpha = 0,$$

$$u^\alpha \nabla_\alpha u_\beta = \phi_\beta, \tag{A.5}$$

$$\partial_\alpha u^\alpha = 0. \tag{A.6}$$

(A.7)

(A.4) is found by explicitly writing the left-hand side in terms of the Christoffel symbols and using (A.2). (A.5) and (A.6) follow from (A.4), (A.2) and (A.3). Finally, (A.7) is obtained by using u^μ explicitly and the assumption that all quantities are independent of x^k. It immediately follows that, unlike the case for constant u_k examined in [2], u^μ is no longer a timelike Killing vector due to (A.4) and is no longer tangent to a timelike geodesic curve by (A.6).

Moreover, the following identities regarding the Christoffel symbols are useful in the calculation of the Ricci tensor:

$$\gamma^{\nu}{}_{k\alpha} = 0, \tag{A.8}$$

$$u^\mu \gamma^{\nu}{}_{\mu\alpha} = 0, \tag{A.9}$$

$$C^{\nu}{}_{\nu\mu} = \phi_\mu, \tag{A.10}$$

$$\Gamma^{\nu}{}_{\nu\mu} = \phi_\mu + \gamma^{\nu}{}_{\nu\mu} \quad .$$

(A.11)

(A.8) and (A.9) again result from the independence of all quantities from x^k and the forms of u^μ and $h_{k\mu}$. Since $\sqrt{-g} = |u_k|\sqrt{|h|}$, one finally gets (A.10) and (A.11) by the definition of φ.

With the help of these identities, the Ricci tensor takes the form

$$R_{\mu\nu} = \hat{r}_{\mu\nu} + \tfrac{1}{2} f_\mu{}^\alpha f_{\nu\alpha} + \tfrac{1}{4} f^2 u_\mu u_\nu - \nabla_\mu \nabla_\nu \phi - \tfrac{1}{2}\phi_\mu \phi_\nu + \tfrac{1}{2}(u_\mu j_\nu + u_\nu j_\mu) \tag{A.12}$$

where we use $f^2 \equiv f^{\alpha\beta} f_{\alpha\beta}$, $j_\mu \equiv f^{\alpha}{}_{\mu|\alpha} - \varphi_\alpha u_{\mu|\alpha}$

and $\hat{r}_{\mu\nu}$ denotes the Ricci tensor of $\gamma^{\mu}{}_{\alpha\beta}$. However, for later convenience it is better to convert the covariant derivatives with respect to $\gamma^{\mu}{}_{\alpha\beta}$ in j_μ to covariant derivatives with respect to $\Gamma^{\mu}{}_{\alpha\beta}$. This is achieved by noting that

$$j_\mu = \nabla_\alpha f^\alpha{}_\mu \quad \phi_\alpha f^\alpha{}_\mu \quad \tfrac{1}{2} f^2 u_\mu = \mathrm{e}^\phi \nabla_\alpha\left(\mathrm{e}^{-\phi} f^\alpha{}_\mu\right) \quad \tfrac{1}{2} f^2 u_{\mu -} \quad - \quad -$$

and using this in (A.12), the Ricci tensor now becomes

$$R_{\mu\nu} = \hat{r}_{\mu\nu} + \tfrac{1}{2}u_\mu\, e^{\phi}\nabla_\alpha\left(e^{-\phi} f^{\alpha}{}_{\nu}\right) + \tfrac{1}{2}u_\nu\, e^{\phi}\nabla_\alpha\left(e^{-\phi} f^{\alpha}{}_{\mu}\right) + \tfrac{1}{2} f_\mu{}^{\alpha} f_{\nu\alpha} - \tfrac{1}{4} f^2 u_\mu u_\nu - \nabla_\mu\nabla_\nu\phi - \tfrac{1}{2}\phi_\mu\phi_\nu. \tag{A.13}$$

The Ricci scalar is then given by

$$R = \hat{r} + u_\nu\, e^{\phi}\nabla_\mu(e^{-\phi} f^{\mu\nu}) + \tfrac{3}{4} f^2 - \Box\phi - \tfrac{1}{2}(\nabla\phi)^2, \tag{A.14}$$

where we use

$$\Box\phi \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu\phi, \qquad (\nabla\phi)^2 \equiv g^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi), \qquad f^{\mu\nu} \equiv g^{\mu\alpha}g^{\nu\beta}f_{\alpha\beta},$$

and $\hat{r}$ denotes the Ricci scalar of $\gamma^{\mu}{}_{\alpha\beta}$. Note that

$$\hat{r} = g^{\alpha\beta}\hat{r}_{\alpha\beta} = \bar{h}^{\alpha\beta}\hat{r}_{\alpha\beta}$$

by using $u^\mu = -\frac{1}{u_k}\delta^\mu_k$, (2) and (A.9) in the explicit calculation of $\hat{r}$. Now using (A.13) and (A.14), one obtains (5) in the text.

Appendix B. Getting (7) from (6)

In this appendix, we present the details of the calculations that lead to (7).

The first task is to relate the covariant derivatives $\tilde{\nabla}_\mu$ and ∇_μ to each other. This is given by [3]

$$\tilde{\nabla}_\mu\omega_\nu = \nabla_\mu\omega_\nu - K^{\sigma}{}_{\mu\nu}\omega_\sigma,$$

where ω_μ is any vector field and

$$K^{\sigma}_{\mu\nu} = \frac{1}{2-D}\left(\delta^{\sigma}_{\mu}\nabla_\nu\phi + \delta^{\sigma}_{\nu}\nabla_\mu\phi - g_{\mu\nu}g^{\sigma\alpha}\nabla_\alpha\phi\right)$$

in our case. We now assume that the vector u_μ is conformally invariant with conformal weight zero, i.e. $\tilde{u}_\mu = u_\mu$. Obviously, one also has $\tilde{\nabla}_\mu\phi \equiv \nabla_\mu\phi$ then. Using these and defining $\tilde{f}_{\alpha\beta} = \tilde{\nabla}_\alpha\tilde{u}_\beta - \tilde{\nabla}_\beta\tilde{u}_\alpha$, it readily follows that the 'Maxwell field' $f_{\alpha\beta}$ also has conformal weight equal to 0, i.e. $\tilde{f}_{\alpha\beta} = f_{\alpha\beta}$.

At this point special care has to be given as to which metric is used for raising and lowering indices. To remove any ambiguity, we now assume that any quantity with a tilde on it has all its indices raised and lowered by using the metric $\tilde{g}_{\mu\nu}$ only; thus, for example, one has

$$\tilde{f}^{\mu}{}_{\nu} = \tilde{g}^{\mu\alpha}\tilde{f}_{\alpha\nu} = e^{\frac{2\phi}{D-2}} g^{\mu\alpha} f_{\alpha\nu} = e^{\frac{2\phi}{D-2}} f^{\mu}{}_{\nu}$$

in our case. One then finds the following useful identities all of which follow from the above preliminaries:

$$
\begin{aligned}
f^2 &= e^{\frac{4\phi}{2-D}} \tilde{f}^2, \\
T^f_{\mu\nu} &= e^{\frac{2\phi}{2-D}} \tilde{T}^f_{\mu\nu}, \\
T^\phi_{\mu\nu} &= \tilde{T}^\phi_{\mu\nu}, \\
\nabla_\mu \nabla_\nu \phi &= \tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi + \frac{2}{2-D} \tilde{T}^\phi_{\mu\nu} \\
g_{\mu\nu} \Box \phi &= \tilde{g}_{\mu\nu} (\Box\phi + (\tilde{\nabla}\phi)^2), \\
g_{\mu\nu} g^{\alpha\beta} &= \tilde{g}_{\mu\nu} \tilde{g}^{\alpha\beta}, \qquad ,
\end{aligned}
$$

$$
\begin{aligned}
e^\phi \nabla_\mu \left(e^{-\phi} f^\mu{}_\nu\right) &= \tilde{\nabla}_\mu \left(e^{\frac{2\phi}{2-D}} \tilde{f}^\mu{}_\nu\right) \qquad , \\
u_\nu e^\phi \nabla_\mu (e^{-\phi} f^{\mu\nu}) &= \tilde{u}_\nu e^{\frac{2\phi}{2-D}} \tilde{\nabla}_\mu \left(e^{\frac{2\phi}{2-D}} \tilde{f}^{\mu\nu}\right).
\end{aligned}
$$

Using these in (6) and arranging the resulting terms carefully, one obtains the Einstein tensor given in (7) which now only involves terms associated with the conformally transformed metric (or the string metric) $\tilde{g}_{\mu\nu}$.

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