ISSN 1054-660X, Laser Physics, 2007, Vol. 17, No. 2, pp. 62–65.

General Approach to Quantum Entanglement

A. A. Klyachko and A. S. Shumovsky*

Faculty of Science, Bilkent University, Bilkent, Ankara, 06800 Turkey

*e-mail: shumo@fen.bilkent.edu.tr

Received July 26, 2006

Abstract—We examine quantum entanglement as a physical phenomenon independent of specific problems of quantum information technologies. Within the dynamic symmetry approach, we briefly discuss the role of quantum fluctuations in formation of entangled states, including single-particle entanglement, relativity of entanglement with respect to the choice of basic observables, and stabilization of robust entanglement.

PACS numbers: 03.67.Mn, 03.65.Ud, 03.67.-a

DOI: 10.1134/S1054660X07020028

Today, quantum entanglement is successfully used in practical realizations of quantum information processing, in particular, in quantum key distribution [1–3]. Nevertheless, revealing the physical nature of this phenomenon and its very definition still remains an important problem.

As a physical phenomenon, quantum entanglement strongly exceeds the narrow bonds of quantum information. It may be observed in systems of different physical natures, from condensed matter to elementary particles. Therefore, it should be studied as a physical phenomenon in quite general settings.

One of the most fruitful methods in quantum physics that has been initiated in the pioneering works by Wigner [4] consists in the analysis of symmetry properties of quantum systems. The approach has been used for decades in quantum mechanics and field theory and has demonstrated very high efficiency.

In the spirit of this ideology, we have developed an approach to quantum entanglement that associates the symmetry properties of a quantum system with specific choice of observables and the result of their measurement (see [5–9] and references therein). Below in this note, we briefly discuss this approach and the most important results obtained within it.

Like any other physical phenomenon, entanglement manifests itself in terms of measurement of physical observables. According to von Neumann’s theory of quantum measurements [10], all physical observables (Hermitian operators) are supposed to be equally accessible. This is not the case for entanglement, in which the parties of a multipartite system can be separated by quite a large distance, so that only *local* measurements are allowed. Another restriction on the available measurements can be caused by the selection rules depending on the type of statistics (either Bose or Fermi).

Thus, adequate description of quantum entanglement requires a certain exceeding of the limits of von Neumann’s theory of quantum measurements. Namely, the definition of a quantum mechanical system should include a specification of both the Hilbert space of states and the available observables [11].

Within our approach, the *basic observables* X_i are defined as the basis of a Lie algebra $\mathcal{L}$ such that the corresponding Lie group $G = \exp(i\mathcal{L})$ defines the *dynamic symmetry* of the system.

This specification of measurements results in the *relativity of entanglement* with respect to the choice of basic observables. The point is that the dynamic symmetry group may have a subgroup $G' \subset G$. In this case, the basic observables $X_i' \in \mathcal{L}'$ and $X_i \in \mathcal{L}$ act in the same Hilbert space $\mathcal{H}$. At the same time, they determine different physical systems.

An example is provided by the three-dimensional system ($\dim\mathcal{H} = 3$) which is usually associated with realization of quantum ternary logic [12, 13]. The dynamic symmetry $SU(\mathcal{H}) = SU(3)$ corresponds to an utter qutrit. The basis of the corresponding Lie algebra $\mathcal{L} = su(3)$ consists of eight independent basic observables (Gell-Mann matrices) [14]. In turn, the dynamic symmetry subgroup $G' = SU(2) \subset SU(3)$ corresponds to the Lie algebra $\mathcal{L}' = su(2)$, whose basis contains only three independent observables (“spin-1” operators). If a state $\psi \in \mathcal{H}$ is entangled with respect to $su(3)$ basic observables, it is also entangled with respect to $su(2)$ basic observables but not vice versa [15].

According to the results [16, 17], entangled states ψ_E form a special class in $\mathcal{H}$. All these states can be obtained from completely entangled states ψ_{CE} by means of certain local operations (usually called SLOCC—stochastic local operations assisted by classical communications):

$$\psi_E = g_l^c \psi_{CE}, \quad g_l^c \in G^c = \exp(\mathcal{L} \otimes \mathbb{C}), \tag{1}$$

where G^c denotes the complexified dynamic symmetry group. All other classes of states (in particular, unentangled states $\psi_{UE} \in \mathcal{H}$) are SLOCC-nonequivalent to ψ_E.

Thus, it is enough to find CE states of a given system. The latter are defined by the following condition [5, 9, 18]:

$$\forall X_i \in \text{Basis}\,\mathcal{L} \quad \langle\psi_{CE}|X_i|\psi_{CE}\rangle = 0. \quad (2)$$

In particular, this means that the vector

$$X_\psi = \sum_i \langle\psi|X_i|\psi\rangle X_i \quad (3)$$

in the Lie algebra $\mathcal{L}$ has zero length if $\psi = \psi_{CE}$:

$$\langle\psi_{CE}|X_{\psi_{CE}}|\psi_{CE}\rangle = 0. \quad (4)$$

It is also possible to show that this vector X_ψ has the maximal length in the case of unentangled states that can be associated with generalized coherent states [5, 19] (concerning generalized coherent states; see [20]).

Either the quadratic form

$$\langle\psi|X_\psi|\psi\rangle = \sum_i \langle\psi|X_i|\psi\rangle^2 \quad (5)$$

itself or a deviation of (5) from its maximal value

$$\Delta(\psi) = \langle\psi_{UE}|X_{\psi_{UE}}|\psi_{UE}\rangle - \langle\psi|X_\psi|\psi\rangle \quad (6)$$

can be used as the measure of the amount of entanglement carried by the state ψ [21, 22]. In particular, the concurrence—a measure of bipartite entanglement introduced by Wootters [23] (also see [24])—has the form

$$C(\psi) = \sqrt{\frac{\Delta(\psi)}{\langle\psi_{UE}|X_{\psi_{UE}}|\psi_{UE}\rangle}} \quad (7)$$

for bipartite systems of any dimension d ($d = \dim\mathcal{H}$) [22].

The quantity (5) has a simple physical meaning. For example, in the case of two qubits, which is important for applications, provided by the polarization of photon twins created by type II down-conversion [25], the basic observables can be associated with the Stokes operators

$$X_1 \sim (a_H^+ a_V + a_V^+ a_H)/\sqrt{2},$$

$$X_2 \sim i(a_H^+ a_V - a_V^+ a_H)/\sqrt{2},$$

$$X_3 \sim a_H^+ a_H - a_V^+ a_V,$$

so that the measurement of concurrence (7) assumes the measurement of three Stokes operators for either outgoing photon beam. The polarization of photons is known to be measured by means of either standard six-state or a minimal four-state ellipsometer [26].

For a long time, we have favored interpretation of quantum entanglement as a manifestation of quantum fluctuations at their extreme [5–9, 18]. This fact immediately follows from the condition of CE (2), which is also expressed by Eq. (4). Together with the quadratic form (5), we can consider the total variance (or the total amount of uncertainty) of basic observables in a state $\psi \in \mathcal{H}$:

$$V(\psi) = \sum_i (\langle\psi|X_i^2|\psi\rangle - \langle\psi|X_i|\psi\rangle^2) \geq 0. \quad (8)$$

Taking into account that the first term in (8) coincides with the Casimir operator (c-number)

$$C_{\mathcal{H}} \equiv \sum_i X_i^2$$

and that the second term is given by the quadratic form (5), we can rewrite Eq. (8) as follows:

$$V(\psi) = C_{\mathcal{H}} - \langle\psi|X_\psi|\psi\rangle. \quad (9)$$

Thus, under the condition of CE (4), total variance (9) achieves its maximal value provided by the Casimir number. This means that CE states manifest the maximum amount of quantum fluctuations of basic observables.

Now note that the existence of quantum fluctuations is a characteristic trait of quantum mechanics that reflects the statistical nature of quantum states and the very definition of quantum observables. A number of quantum phenomena like coherence and squeezing are defined in terms of quantum fluctuations in the corresponding states. The almost classical generalized coherent states manifest a minimum level of quantum fluctuations. In turn, CE states form an opposite pole of entirely quantum states. In a sense, they correspond to the generalized squeezed states, at least for certain dynamical symmetries [27] (concerning generalized squeezed states, see [28]).

Note also that the quantum uncertainty describing the measurement of an observable X_i in a given pure state ψ was interpreted by Wigner an Yanase [29] as the amount of specific quantum information about the state ψ that can be extracted from *macroscopic* measurement of this observable X_i. Thus, the total uncertainty (9) can also be associated with the total amount of quantum information about the state ψ.

Anther logical corollary of the definition of entangled (1) and CE (2) states is that these conditions do not assume a multipartite character of the system. In other words, entanglement can exist for a single particle with respect to its intrinsic degrees of freedom [15]. An example of some physical interest is provided by a single "spin-1" particle, i.e., by a system with dynamic symmetry $G = SU(2)$ and basic observables forming the basis of $\mathcal{L} = su(2)$ in three dimensions (the so-called $su(2)$ qutrit). The existence of entanglement in such a system immediately follows from the Clebsch–Gordon decomposition of the two-qubit (two "spin-1/2") space of states

$$\mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}_{\frac{1}{2}} = \mathcal{H}_1 \oplus \mathcal{H}_0.$$

Here the single "spin-1" space $\mathcal{H}_1$ is spanned by the symmetric triplet. If we choose the basis of $\mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}_{\frac{1}{2}}$ as

$$|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \quad (10)$$

the first three states just form the symmetric triplet that can be associated with the spin-projection states $|+1\rangle$, $|0\rangle$, $|-1\rangle$, respectively. The last (antisymmetric) state in (10) corresponds to the scalar space $\mathcal{H}_0$. Thus, states of a single "spin-1" object are equivalent to the states of two qubits in the symmetric sector of the Hilbert space. It can be easily seen that the state $|0\rangle$ associated with the CE two-qubit state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ can be interpreted as a completely entangled one with respect to intrinsic (qubit) degrees of freedom.

Such a "spin-1" system can be either bipartite or local (single-particle). A three-level atom and biphoton [30] represent important examples of the former case. In the case of a biphoton, qubits are formed by the polarization of two photons. In the states $|\pm1\rangle$, the biphoton has a given polarization, while in the CE state $|0\rangle$ it is unpolarized. In the case of the three-level atom, qubits are formed by a transition between the common level and two other levels.

An example of the $SU(2)$ system in three dimensions is given by the isospin 1, which describes the isotriplet of π mesons [31]. In this case, the states $|\pm\rangle$ correspond to the particles $\pi^{\pm}$ with either positive or negative charge. In turn, the state $|0\rangle$ corresponds to the neutral π^0 meson. The qubits can be considered as completely intrinsic degrees of freedom. Namely, the fundamental representation of the isospin-1 symmetry is given by two quark doublets, consisting of "up" (u) and "down" (d) quarks and antiquarks $\bar{u}$, $\bar{d}$ [32]. Each doublet can be naturally interpreted as a qubit. The state $|0\rangle$ has the structure

$$|0\rangle = \pi^0 = (u\bar{u} + d\bar{d})/\sqrt{2}$$

and is hence completely entangled with respect to quark degrees of freedom. Two charged π mesons correspond to the UE states of quarks.

Physical interpretation of the above result is based on the definition of CE states as the states with maximum amount of quantum fluctuations. Because of this, the π^0 meson should be much less stable than $\pi^{\pm}$, which agrees with the experimental data.

A more optical example of single-particle entanglement is provided by an electric dipole (E1) photon emitted by the E1 atomic transition, whose *total angular momentum* $j = 1$ and whose spin (s) and orbital momentum (l) can be associated with two qubits [33]: the two helicities and the two allowed values $l = 0, 2$. Since the vector potential of the single E1 photon cannot be factorized with respect to spin and orbital parts [34], its state can be interpreted as entangled with respect to intrinsic degrees of freedom—helicity and orbital angular momentum. For some other physical examples, see [15, 27].

One more corollary contains an idea of how to make robust entanglement. The point is that practical implementations need stable or metastable states with a high amount of entanglement. In view of our result, robust entanglement can be achieved in the following way. First, it is necessary to prepare a state with the maximum amount of quantum fluctuations. Then, the energy of the system should be decreased to a minimum (or local minimum) under the condition of preservation of the level of quantum fluctuations. Realization of this scheme in three-level atoms with the Λ-type transition in the cavity, discarding Stokes photons by means of either absorption or leakage, has been discussed in [35–38]. It is also interesting that, in the absence of cavity, entanglement two two-level atoms can be stabilized by a classical driving field [39, 40].

Summarizing, we want to emphasize that the dynamic symmetry approach to quantum entanglement reveals it physical nature as a manifestation of quantum uncertainties at their extreme independent of whether the system is composite or not. In particular, it is capable of defining single-particle entanglement with respect to intrinsic degrees of freedom. Such a single-particle entanglement has nothing in common with the so-called single-photon entanglement [41], where an external "geometrical" qubit generated by a beam splitter is involved.

It is possible to say that, ideologically, the dynamic symmetry approach allows us to separate the essential from the accidental in the definition of quantum entanglement. In particular, nonseparability of states, nonlocality of a quantum system, and violation of Bell's conditions do not form sufficient conditions of entanglement and hence cannot be used as a definition of this phenomenon.

The analysis of dynamic symmetry allows us to establish the relativity of entanglement with respect to the choice of basic observables and to specify the minimal umber of measurements required to measure the amount of entanglement. In particular, for a bipartite system, the measures of entanglement (6) and (7) have the same value for both parties and hence can be measured locally [22]. The number of measurements is then restricted by the number of local basic observables. In the simplest case of two qubits, for example, the three observables (Pauli operators) should be measured for either party. In the case of photon twins, this means the measurement of three Stokes parameters. This agrees with an early result [42].

The revelation of the role of quantum fluctuations in the formation of quantum entanglement also opens the way for stabilization of entanglement in physical systems by means of a certain mini-max procedure: minimum energy of the system at maximum amount of uncertainty.

So far, we have considered pure entangled states. The generalization of the approach to the case of mixed states meets certain complications. The point is that the density matrix contains classical fluctuations caused by the statistical nature of the state together with quantum fluctuations. Their separation represents a hard problem of extremely high importance. One of the possible approaches consists in the use of the methods of thermo-field dynamics [43], which allows a mixed state to be represented in terms of a pure state of doubled dimension.

REFERENCES

1. L. Marcikic, H. de Riedmatten, W. Tittel, and N. Gisin, Nature (London) **42**, 509 (2003).
2. A. Poppe, A. Fedrizzi, R. Ursin, et al., Opt. Express **12**, 3865 (2004).
3. J. Ouelette, Industrial Phys. **10**, 22 (2004).
4. E. P. Wigner, Ann. Math. **40**, 149 (1939).
5. A. A. Klyachko, quant-ph/0206012.
6. A. A. Klyachko and A. S. Shumovsky, J. Opt. B: Quantum Semiclass. Opt. **5**, S322 (2003).
7. A. A. Klyachko and A. S. Shumovsky, J. Opt. B: Quantum Semiclass. Opt. **6**, S29 (2004).
8. A. A. Klyachko and A. S. Shumovsky, Laser Phys. **14**, 134 (2004).
9. A. A. Klyachko and A. S. Shumovsky, J. Phys.: Conf. Ser. **36**, 87 (2006).
10. J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princetown Univ. Press, Princetown, 1996).
11. R. Hermann, *Lie Groups for Physicists* (Benjamin, New York, 1966).
12. H. Bechman-Pasquinucci and A. Peres, Phys. Rev. Lett. **85**, 3313 (2000).
13. G. A. Maslennikov, A. A. Zhukov, M. V. Chekhova, and S. P. Kulik, J. Opt. B: Quantum Semicalss. Opt. **5**, 530 (2003).
14. C. M. Caves and G. J. Milburn, Opt. Commun. **179**, 439 (2000).
15. M. A. Can, A. A. Klyachko, and A. S. Shumovsky, J. Opt. B: Quantum Semiclass. Opt. **7**, L1 (2005).
16. W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).
17. F. Verstraete, J. Dehaene, and B. De Moor, Phys. Rev. A **68**, 012103 (2003).
18. M. A. Can, A. A. Klyachko, and A. S. Shumovsky, Phys. Rev. A **66**, 02111 (2002).
19. H. Barnum, E. Knill, G. Ortiz, and L. Viola, Phys. Rev. A **68**, 032308 (2003); A. M. Perelomov, Commun. Math. Phys. **26**, 222 (1972).
20. A. M. Perelomov, *Generalized Coherent States and Their Applications* (Springer, Berlin, 1986).
21. A. A. Klyachko, B. Öztop, and A. S. Shumovsky, Appl. Phys. Lett. **88**, 124102 (2006).
22. A. A. Klyachko, B. Öztop, and A. S. Shumovsky, quant-ph/0606119.
23. W. K. Wooters, Phys. Rev. Lett. **80**, 2245 (1998).
24. P. Rungta, V. Bužek, C.M. Caves, et al., Phys. Rev. A **64**, 042315 (2001).
25. B. E. Abouraddy, B. E. A. Saleh, A. V. Sergienko, and M. C. Teich, Phys. Rev. A **64**, 050101 (2001); M. B. Nasr, A. F. Abouraddy, M. C. Booth, et al., Phys. Rev. A **65**, 023816 (2002); G. DiGiuseppe, M. Atature, M. Shaw, et al., Phys. Rev. A **68**, 063817 (2003).
26. J. Řeháček, B. C. Englert, and D. Kaszlikowski, Phys. Rev. A **70**, 052321 (2004).
27. S. Binicioğlu, M. A. Can, A. A. Klyachko, and A.S. Shumovsky, Citations quant-ph/0604182 (April 2006).
28. M. Kitagawa and M. Ueda, Phys. Rev. A **47**, 5138 (1993).
29. E. P. Wigner and M. M. Yanase, Proc. Natl. Acad. Sci. USA **19**, 910 (1963).
30. A. V. Burlakov, M. V. Chechova, O. A. Karabutova, et al., Phys. Rev. A **60**, R4209 (1999); Yu. I. Bogdanov, M. V. Chekhova, S. P. Kulik, et al., Phys. Rev. Lett. **93**, 230503 (2004).
31. N. N. Bogoliubov and D. V. Shirkov, *Quantum Fields* (Benjamin/Cummings, Reading, MA, 1982).
32. K. Huang, *Quarks, Leptons, and Guage Fields* (World Sci., Singapore, 1982).
33. M. A. Can, Ö. Çakir, A. Horzela, et al., Laser Phys. **14**, 1180 (2004).
34. V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electrodynamics* (Nauka, Moscow, 1989; Pergamon, Oxford, 1982).
35. M. A. Can, A. A. Klyachko, and A. S. Shumovsky, Appl. Phys. Lett. **81**, 5072 (2002).
36. M. A. Can, Ö. Çakir, A. A. Klyachko, and A. S. Shumovsky, Phys. Rev. A **68**, 022395 (2003).
37. A. Biswas and G. S. Agarwal, J. Mod. Opt. **51**, 1627 (2004).
38. Ö. Çakir, H. T. Dung, L. Knöll, and D.-G. Welsch, Phys. Rev. A **71**, 032326 (2005).
39. Ö. Çakir, A. A. Klyachko, and A. S. Shumovsky, Appl. Phys. Lett. **86**, 174103 (2005).
40. Ö. Çakir, A. A. Klyachko, and A. S. Shumovsky, Phys. Rev. A **71**, 034303 (2005).
41. S. M. Tan, D. F. Walls, and M. J. Collett, Phys. Rev. Lett. **66**, 252 (1991); L. Hardy, Phys. Rev. Lett. **73**, 2279 (199-4); P. G. Kwiat and R. J. Hughes, Science **289**, 1421 (2000).
42. J. M. G. Sancho and S. F. Huelga, Phys. Rev. A **61**, 042303 (2000).
43. Y. Takahashi and H. Umezawa, Int. J. Mod. Phys. B **16**, 1755 (1996).