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Boundary control of a rotating shear beam with observer feedback

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Abstract

We consider a flexible structure modeled as a shear beam which is free to rotate on the horizontal plane. We first model the system by using partial differential equations and we propose boundary feedback laws to achieve set-point regulation of the rotation angle as well as to suppress elastic vibrations. The main advantage of the proposed design, which consists of a decoupling controller together with an observer, is that it is easy to implement. We utilize a coordinate transformation based on an invertible integral transformation by using Volterra form and backstepping techniques. We show that with the proposed controller, the control objectives are satisfied.

Keywords

Backstepping, boundary control, distributed parameter systems, flexible structures, shear beam, Volterra transformation

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1. Introduction

The progress in the construction of various mechanical structures, e.g. in robotics and space structures on the macroscopic scale, as well as micro-machines, atomic force microscopy, etc. on the microscopic scale, necessitates the use of lightweight materials for various practical reasons (Doğan and Morgül, 2010). Such materials usually exhibit flexural vibrations, and to model such structures one has to use Partial Differential Equations (PDEs). In practice, when designing controllers for such systems, usually these PDE models are reduced to Ordinary Differential Equations (ODEs) using various methods such as model reduction, finite element analysis, discretization, etc. (Doğan and Istefanopulos, 2007). However, such ODE models have some drawbacks and usually controllers designed using these ODE models limit the performance of such systems (Doğan, 2006).

One of the most frequently encountered of these flexible mechanical structures is the flexible beam; these are typically used to model flexible links of robotic arms, tips of atomic force microscopes, etc. Among the various advantages of using flexible beams, the main ones are their light weight and low energy consumption. There are various PDE models for flexible beams such as Euler–Bernoulli, Rayleigh, Timoshenko beam equations. A comparison of these beam models can be found in Baruh (1999, Section 11.2). Among these, the most advanced and comprehensive one is the Timoshenko beam model (Morgül, 1992; Baruh, 1999). This model, under the "slender beam" assumption, can alternatively be represented as a shear beam (Baruh, 1999; Meirovitch, 2001).

Various methods have been proposed for control of flexible links in the literature (Kim and Renardy, 1987; Morgül, 1991, 1992; Luo, 1993; Luo et al., 1999; Guo, 2002; Wang and Gao, 2003). Recently, in Smyshlyaev and Krstic (2004, 2005) and Krstic et al. (2006), a structural approach to PDEs with backstepping techniques was proposed.

In this paper, we consider a shear beam clamped to a rigid body at one end and free at the other end. The whole configuration is free to rotate on the horizontal plane. We first give the equations of motion for such system. We will use a PDE model for the flexible beam without resorting to reducing the resulting equations to an ODE model. Our control objective is to rotate the flexible beam to a desired angle and suppress the flexural vibrations. We use a technique first introduced in Smyshlyaev and Krstic (2004, 2005) and Krstic et al. (2006) to transform the system equations to another

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set of equations which has well-known stability properties. This transformation is done by using an invertible integral operator (Volterra type) with a smooth kernel. We give the analytical expression of such a kernel. In the process of obtaining an appropriate kernel function, we also obtain an appropriate boundary control law to stabilize the closed-loop system. We also present some simulation results which confirm the stability of the closed-loop system. Finally, we give some concluding remarks.

2. Analytical model

We consider a flexible structure consisting of a flexible (shear) beam which is clamped to a rigid body at one end, free at the other, as shown in Figure 1. Referring to Figure 1, the various symbols represent the following: X_o, Y_o: global inertial system of coordinates; X_1, Y_1: body-fixed system of coordinates attached to undeformed beam; θ_1: angular displacement of beam; u: flexural displacement of beam.

In this section, starting with the Timoshenko beam model, the partial differential equations with boundary conditions are derived using the Volterra state transformation (Porter, 1990; Krstic et al., 2006). The link is modeled in clamped-free configuration, since natural modes of the separated clamped-free links agree very well with actual ones compared to pinned-free configuration (Hastings and Book, 1987). Assuming the manipulator rotates in the horizontal plane, in the absence of gravity the potential energy depends only on the flexural deflections.

The equations of motion for a Timoshenko beam can be given, with the notation in Table 1, as follows:

$$\epsilon\,\ddot{u} = b\,u_{xx} - \alpha_x - \epsilon\,x\,\ddot{\theta}_1, \tag{1}$$

$$\mu\epsilon\,\ddot{\alpha} = \epsilon\,\alpha_{xx} - \alpha + b\,u_x, \tag{2}$$

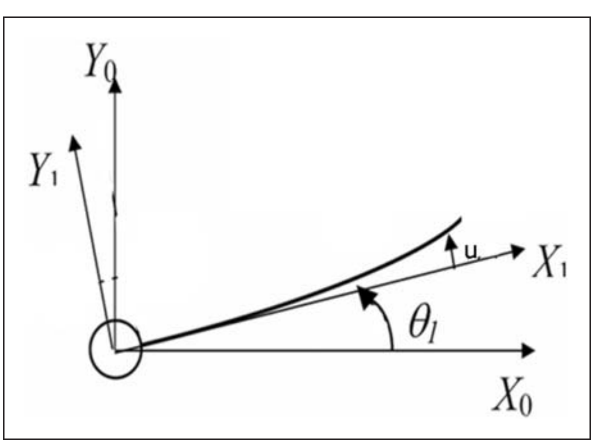

Figure 1. Beam configuration.

where a dot represents time derivative, a subscript as in u_x denotes the spatial derivative with respect to x, for $0 \le x \le L$ and $t \ge 0$. Here, EI denotes bending stiffness, b is defined as $b = EI/\rho$, EA denotes axial stiffness, μ is defined as $\mu = \rho/EA$, and the shear coefficient, ϵ, is a linear function of EI/GA (Sievers et al., 1988; Reddy, 1993). Since the ratio between the length of the beam and its thickness is sufficiently large, we can take $\mu = 0$ approximately. Then the slender beam can easily be modeled as a shear beam (Reddy, 1993; Meirovitch, 2001). Thus, the governing equations for a rotating shear beam can be given below:

$$\epsilon\,\ddot{u} = b\,u_{xx} - \alpha_x - \epsilon\,x\,\ddot{\theta}_1, \tag{3}$$

$$0 = \epsilon\,\alpha_{xx} - \alpha + b\,u_x. \tag{4}$$

By differentiating (3) with respect to x and adding to (4), we obtain:

$$\epsilon\,\ddot{u}_x - b\,u_{xxx} + \alpha_{xx} + \epsilon\,\ddot{\theta}_1 + \epsilon\,\alpha_{xx} - \alpha + b\,u_x = 0. \tag{5}$$

Now if we differentiate (5) with respect to x and subtract $1/\epsilon$ times (3) then we have

$$\begin{aligned} &\epsilon\,\ddot{u}_{xx} - b\,u_{xxxx} + (1+\epsilon)\alpha_{xxx} + b\,u_{xx}, \\ &-\alpha_x - \ddot{u} + \frac{b}{\epsilon}u_{xx} - \frac{1}{\epsilon}\alpha_x - x\ddot{\theta}_1 = 0. \end{aligned} \tag{6}$$

Finally, by calculation of α_{xxx} from (4), we obtain the following shear beam model as a single second-order-in-time, fourth-order-in-space PDE:

$$\ddot{u} - \epsilon\,\ddot{u}_{xx} + b\,u_{xxxx} = -x\,\ddot{\theta}_1. \tag{7}$$

Table 1. Parameters for PDE model

Parameter	Description
E	Young's Modulus
G	Shear Modulus
A	Cross-sectional area
I	Cross-sectional area moment
I_h	Inertia of the hub
L	Length of the beam
x	Coordinate along the axial center
$u(x, t)$	Transverse movement
$\alpha(x, t)$	Angle of distortion due to shear
$\dot{u}(x, t)$	Time rate of transverse movement
$u_x(x, t)$	Axial rate of transverse movement
ρ	Linear density
τ_1	Input torque at base motor
θ_1	Angular position of the beam due to rotation

The usual clamped-free boundary conditions for the shear beam can be given as follows (Meirovitch, 1967):

$$u(0,t) = 0, \quad u_x(0,t) = \alpha(0,t), \tag{8}$$

$$u_{xx}(L,t) = 0, \quad \mathbf{u_{xxx}(L,t) = 0}. \tag{9}$$

Note that by using (4) we can obtain $\alpha(x, t)$ in terms of $u(x, t)$. This could be done in various ways, e.g. by using standard transformation techniques (Krstic et al., 2006). After straightforward calculations, we obtain the following:

$$\alpha_x(x) = -a^2 c \int_0^x \sinh(cx - cy) u(y) dy - a^2 u(x), \tag{10}$$

where $a^2 = b/\epsilon$ and $c^2 = 1/\epsilon$. Note that for notational simplicity, from now on we will use the notation $\boldsymbol{\alpha}(\mathbf{x})$ and $\mathbf{u}(\mathbf{x})$ instead of $\boldsymbol{\alpha}(\mathbf{x}, \mathbf{t})$, $\mathbf{u}(\mathbf{x}, \mathbf{t})$. By substituting (10) into (3), the governing equations (3) and (4) can be simplified as a single equation given below. For the rigid body rotation, by applying conservation of momentum at the base, we obtain the open-loop system as follows:

$$\epsilon \ddot{u} = b u_{xx} + a^2 u + a^2 c \int_0^x \sinh(cx - cy) u(y) dy - \epsilon x \ddot{\theta}_1, \tag{11}$$

$$I_h \ddot{\theta}_1 - EI u_{xx}(0,t) = \tau_1. \tag{12}$$

Note that the last equation is frequently used in the literature; see Luo (1993), or Equation (4.10) in Luo et al. (1999, Section 4.1).

Remark 2.1 *It is true that flexible beams naturally have various types of damping, e.g. Kelvin–Voigt, viscous, etc. However, from a mathematical point of view, establishing stability of the controlled system without using a damping term is a more challenging and interesting problem. Intuitively, if one includes such damping terms, then a similar analysis could be repeated. The existence of such an internal damping term will naturally enhance the stability of the closed-loop system.*

3. Controller design

The controller for the rigid part can be designed to provide exponential decaying of derivatives and to achieve the desired set point such that

$$\tau_1 = -EI u_{xx}(0,t) - k_1 \dot{\theta}_1 - k_2(\theta_1 - \theta_d), \tag{13}$$

where k_1, k_2 are positive constants and θ_d is the constant desired position. To simplify the flexible equation (11), we first define a new state variable $w(\cdot)$ by applying an invertible Volterra state transformation with smooth kernel $k(x, y)$:

$$w(x) = u(x) - \int_0^x k(x,y) u(y) dy. \tag{14}$$

For the rationale and methodology behind using such a transformation, see Porter (1990); Krstic et al. (2006). The kernel $k(x, y)$ should be chosen so that the resulting equations in terms of the transformed variable $w(x)$ have nice stability properties. Such a resulting system can be given as follows:

$$\epsilon \ddot{w} = b(w_{xx} - e w), \tag{15}$$

$$w_x(L) = -c_o \dot{w}(L), \tag{16}$$

$$w_x(0) = c_1 (\epsilon/b) \ddot{\theta}_1, \tag{17}$$

where e, c_o, c_1 are positive controller gains. Note that the boundary condition (17) is of crucial importance to obtain Equations (10) and (11); see also Remark 3.2. Hence, the open-loop system given by (11) and (12) can be transformed into the closed-loop system given by (15)–(17) with the following boundary control law:

$$\mathbf{u_x(L)} = \int_0^L k_x(L,y) u(y) dy + k(L,L)\mathbf{u(L)} - c_o \dot{\mathbf{u}}(L) + c_o \int_0^L k(L,y) \dot{u}(y) dy. \tag{18}$$

Obviously we need to describe the kernel $k(x, y)$ or its properties at this stage. In addition to Equations (11)–(17), the following equations are also required to obtain the control law (18) and some conditions for the kernel:

$$w_x(x) = u_x(x) - \int_0^x k_x(x,y) u(y) dy - u(x) k(x,x) \tag{19}$$

$$w_{xx}(x) = u_{xx}(x) - u(x)[k_x(x,x) + k_y(x,x)] - u_x(x) k(x,x) - u(x) k_x(x,x) - \int_0^x k_{xx}(x,y) u(y) dy \tag{20}$$

$$\epsilon \ddot{w}(x) = \epsilon \ddot{u}(x) - \int_0^x k(x,y)\, \epsilon \ddot{u}(y) dy \tag{21}$$

By following the methodology introduced in Krstic et al. (2006), the term $\epsilon \ddot{u}(\cdot)$ should be substituted in (21) by using (11). Then, the $\epsilon \ddot{w}(x)$ and $w_{xx}(x)$ obtained from (20) can be replaced in (15). After some calculations, it can be shown that the required kernel $k(x, y)$ should satisfy the nonhomogeneous PDE:

$$k_{yy} - k_{xx} + fk = \epsilon^{-1.5}\sinh(cx - cy) \quad (22)$$

$$k(x, x) = -fx/2 \quad (23)$$

$$k(0, 0) = 0 \quad (24)$$

$$k(x, 0) = x - \int_0^x y\, k(x, y)\, dy \quad (25)$$

$$k_y(x, 0) = 0 \quad (26)$$

where $f = a^2/b + e$. The analytical solution of (22)–(26) for the kernel function $k(x, y)$ can be obtained by using standard methods (Debnath, 1997). After some lengthy calculations, we obtain the following explicit form of the kernel $k(x, y)$:

$$k(x, y) = 0.5[h(0) + h(L)] + 0.5\int_0^L J_0(z)\, g(\xi) \mathrm{d}\xi - 0.5yf\int_0^L (J_1(z)/z)\, h(\xi)\mathrm{d}\xi - m\sinh(cx - cy), \quad (27)$$

where $m = -\epsilon^{-1.5}/f$, $z = \sqrt{(y^2 - (x - \xi)^2)f}$, $J_0(\cdot)$ and $J_1(\cdot)$ are Bessel functions, $h(x) = (m - 0.81)\sinh(cx) + 0.62521x$, and $g(x) = -cm\cosh(cx)$. Note that the functions $h(\cdot)$ and $g(\cdot)$ are generated by boundary conditions of the kernel. For an alternate expression of the kernel, see Appendix A.

Note that the stability of the closed-loop system can be analyzed with Lyapunov stability theory by using the following Lyapunov function candidate (Morgül, 1992):

$$V = d_1(\|w_x\|^2 + e\|w\|^2) + \epsilon\|\dot{w}\|^2 + d_2\epsilon < w, \dot{w} >, \quad (28)$$

where d_1, d_2 are appropriate positive constants, $\|w_x\|^2 = \int_0^L w_x^2\, dx$ is the 2-norm and $< \cdot >$ denotes the usual inner product in $L_2(0, L)$ space. Since the proof of the asymptotic stability of the closed-loop system requires some rigorous definitions and some lengthy calculations, the main part of the proof is given in Appendix B to improve the readability of the paper.

Remark 3.1 *Note that to implement the control law given by (18), we need the measurements of $u(x, t)$, $u(L, t)$ and $\dot{u}(L, t)$. The last two can be measured easily. However, it is not practical to measure $u(x, t)$, but it can be estimated by using an appropriate observer. Following the ideas given in Krstic et al. (2006), such an observer can be designed with the help of the kernel $k(x, y)$ given above as explained in the next section.*

Remark 3.2 *The boundary condition (17) can easily be implemented as the secondary controller due to actuation and measurement at the clamped end. Furthermore, this secondary control law is indispensable in simplifying the open-loop system and in obtaining the analytic kernel solution. This is achieved by decoupling the rigid and flexible coordinates, see (8) and (17), which results in the exponential decay of the solutions of the closed-loop system. The whole structure of the proposed design and its implementation is quite simple and straightforward.*

4. Observer design

Similar to the controller design, to define the observer error dynamics we need an auxiliary dynamics with well-known stability properties. Such a system can be given as follows:

$$\epsilon\,\ddot{\tilde{w}} = b\,(\tilde{w}_{xx} - e_2\,\tilde{w}), \quad (29)$$

$$\tilde{w}_x(L) = -c_2\dot{\tilde{w}}(L), \quad (30)$$

$$\tilde{w}_x(0) = 0, \quad (31)$$

where e_2, c_2 are positive observer gains and observer error is assumed to be zero at the clamped end. After applying the Volterra state transformation the observer error, $\tilde{u}(x)$, can be expressed as follows:

$$\tilde{u}(x) = \tilde{w}(x) - \int_0^x p(x, y)\,\tilde{w}(y)dy. \quad (32)$$

Also, observer error can be defined as $\tilde{u}(x) = u(x) - \hat{u}(x)$ here, and the observer kernel, $p(x, y)$ is the dual version of the controller kernel $k(x, y)$ with appropriate boundary conditions. Hence we are ready to introduce observer dynamics driven by $\tilde{w}(x)$ such that

$$\epsilon\,\ddot{\hat{u}} = b\,\hat{u}_{xx} + a^2\,\hat{u} + a^2\,c\int_0^x \sinh(cx - cy)\,\hat{u}(y)dy - \epsilon\, x\,\ddot{\theta}_1 + p_y(x, L)[\mathbf{u}(\mathbf{L}) - \hat{\mathbf{u}}(\mathbf{L})] - c_2 p(x, L)[\dot{\mathbf{u}}(L) - \dot{\hat{\mathbf{u}}}(\mathbf{L})]. \quad (33)$$

Note that $p_y(x, L)$, $p(x, L)$ are dual counterparts to the controller gains $k_x(L, y)$, $k(L, y)$. The observer dynamics comply with usual observer setup which can be explained as copy of the plant plus error feedback. After some lengthy calculations using the integral transformation (32) and auxiliary dynamics (29), we get the following equations:

$$p_{xx} - p_{yy} + \tilde{f}p = \epsilon^{-1.5}\sinh(cx - cy), \quad (34)$$

$$a^2\,c\int_0^x \sinh(cx - cy)\int_0^y p(y, s)\tilde{w}(s)\,ds\,dy = c_2\,p(x, L)\dot{\tilde{\mathbf{u}}}(\mathbf{L}) - p_y(x, L)\,\tilde{\mathbf{u}}(\mathbf{L}), \quad (35)$$

where $\tilde{f} = a^2/b + e_2$. The analytical solution of (34) for the observer kernel function $p(x, y)$ can be obtained easily by the duality of Equations (22)–(34). Finally, Equation (35) will help us to construct the observer dynamics with the auxiliary one, $\tilde{w}(x)$, in lieu of $\tilde{u}(x)$. Besides, the new control law based on the observer can be given below:

$$\mathbf{u_x}(\mathbf{L}) = \int_0^L k_x(L, y)\hat{u}(y)dy + k(L, L)\mathbf{u}(\mathbf{L}) - c_o\,\dot{\mathbf{u}}(L) + c_o \int_0^L k(L, y)\,\dot{\hat{u}}(y)dy. \quad (36)$$

5. Simulation results

For the simulations, we will use the set of parameters shown in Table 2, which are taken from Doğan (2006).

The proposed control scheme was tested with a simulation program implemented in MATLAB. The PDEs were discretized in the space domain by the finite difference method to obtain ODEs at each of the nodes. Then, the ODEs were solved numerically. Instead of dealing with the complexity of the fourth-order derivative approximation, the second-order derivative approximation has been used by virtue of a coordinate transformation. Those states are more meaningful in a real problem as well since they correspond to physical variables such as deflections, velocity and bending moments. However, the number of ODEs to solve and the computation time are increased in return for the robust stability of the numeric scheme. The explicit finite difference scheme that requires very small time steps and is easy to implement efficiently is adapted from Abhyankar et al. (1993). Chaotic vibrations of a modified Euler–Bernoulli beam, that are difficult to catch, have been solved successfully by the same numeric scheme in Abhyankar et al. (1993). On the other hand, robust numerical stability is achieved by making the ratio $\Delta t\ /\ \Delta x^2 \leq 0.5$ as low as possible. The parameters for the system (11)–(12) are listed in Table 2. The simulation results are presented in Figures 2–7. Note that Figure 4 represents the bending strain of the beam near the free end point $x = L$. Decoupling between the rigid coordinates and flexible ones is achieved by control laws, and is observed during simulations. For θ_1 and $\dot{\theta}_1$, see Figures 5 and 6; the converging properties (two exponential decaying modes for the derivatives) can be adjusted

Table 2. Parameters of the beam

Parameter	Value
Length of the beam	$L = 0.6$ m
Time step	$\Delta t = 1e^{-4}$ s
Spatial step	$\Delta x = L/30$ m
Young's Modulus	$E = 70$ GPa
Density	2742 kgm^{-3}
Thickness of the beam	$t_1 = 0.003175$ m
Height of the beam	$b_o = 0.0654$ m
Shear coefficient	$\epsilon = 1.6801$
Hub inertia	$I_h = 0.0055$ kgm^2
θ_d (desired)	$\pi/3$ rad
Controller gains	$c_o = 59$
	$k_1 = I_h \cdot 600$
	$k_2 = I_h \cdot 800$
	$c_1 = 1$, $e = 0.9$
Observer gains	$c_2 = 82$ and $e_2 = 0.3$

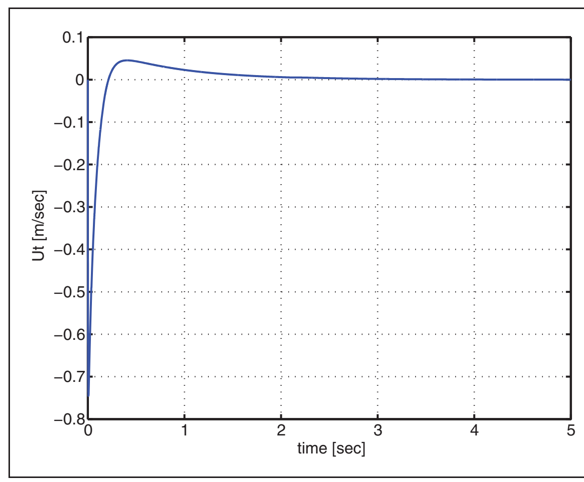

Figure 2. Flexural velocity at the end of the beam.

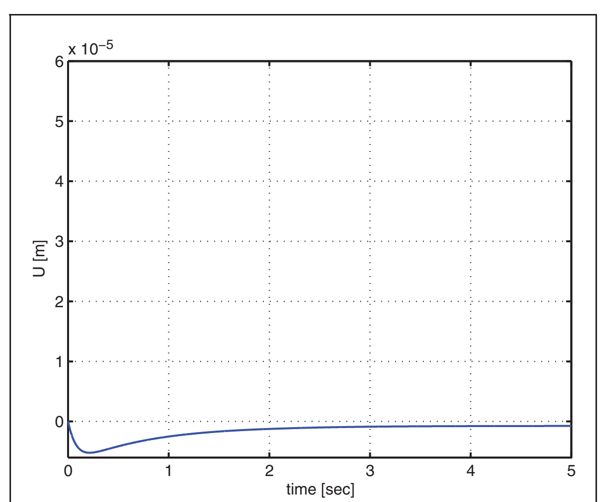

Figure 3. Deflection at the end of the beam.

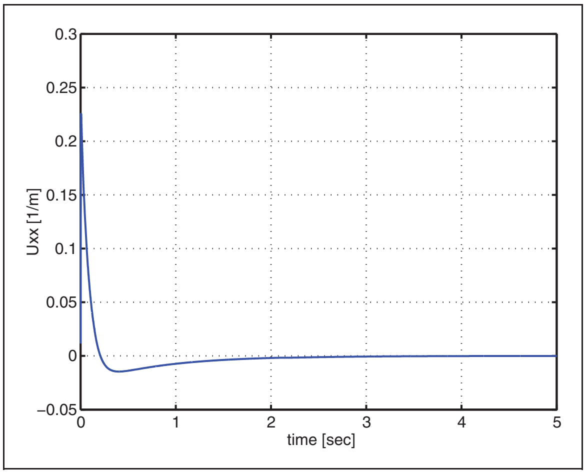

Figure 4. Bending strain of the beam.

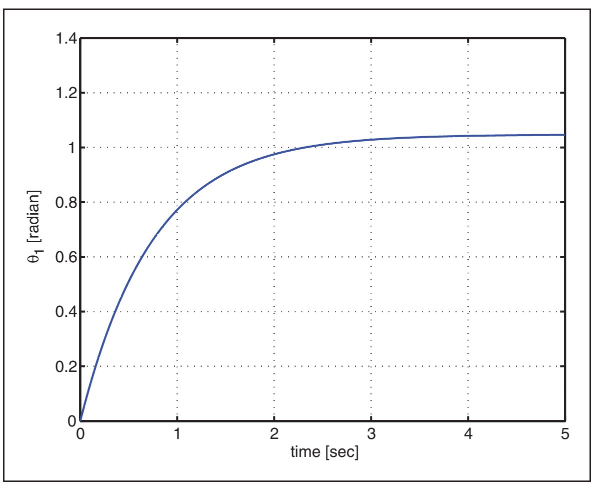

Figure 5. Joint angle of the beam.

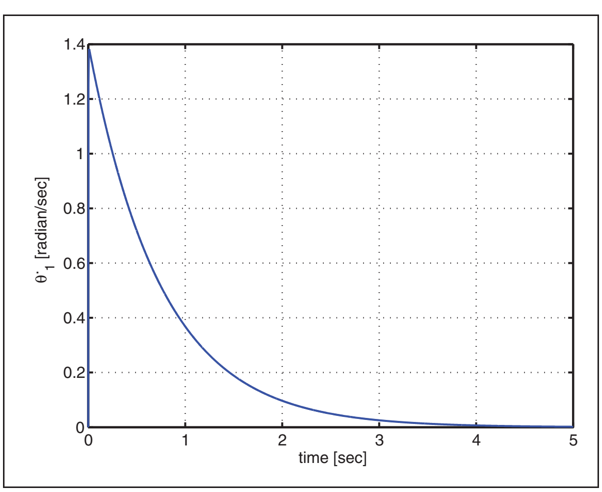

Figure 6. Joint angular velocity of the beam.

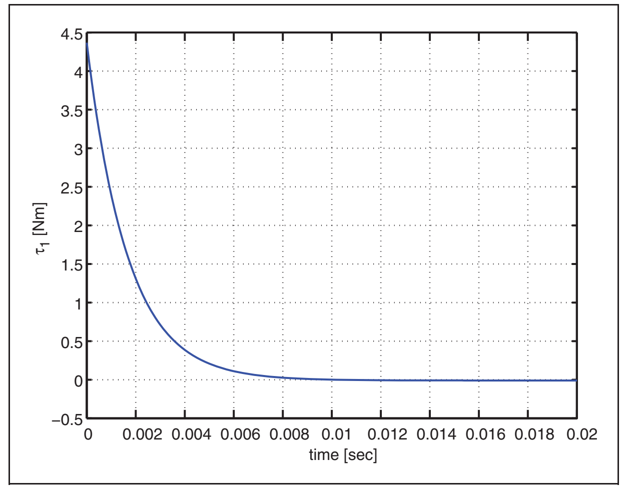

Figure 7. Control torque.

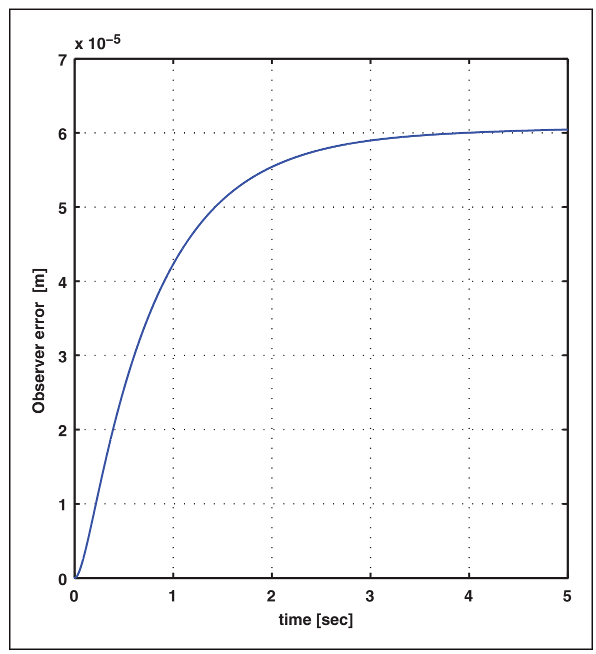

Figure 8. Observer error.

independently with k_1, k_2. Therefore, $\ddot{\theta}_1$ can be used at boundary condition (17) by exponential decaying time history, and also to simplify the closed-loop system. Observer error converged to a reasonable small value which is physically acceptable as well in Figure 8. Note that the observer error is expected to converge to zero, and the small value that we obtained in our simulation is in our opinion due to some numerical errors resulting from our discretization scheme. Fast convergence to zero for observer error rate is quite satisfactory in Figure 9. Finally, smooth time histories of all variables of interest without overshoot show the effectiveness of

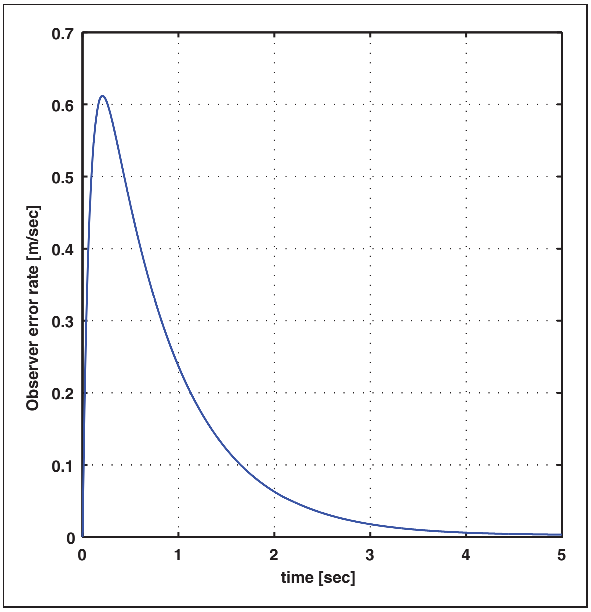

Figure 9. Observer error rate.

the controller performance with relatively low control energy.

6. Conclusions

In this work, a complex beam model is simplified, becoming easy to analyze and to control, and no damping term is used in the system model. We consider the set-point control of a rotating shear beam. We assume that the shear beam is free to rotate on the horizontal plane. In this research, the system equations are first transformed into another set of equations which has well-known stability properties by using an invertible Volterra state transformation. We also obtain the kernel function of such a transformation analytically by virtue of the proposed design. This process is also very efficient in improving the observer dynamics that will give the required boundary control laws. Our simulation results show that the proposed control scheme is effective in both achieving correct orientation and in suppression of flexural vibrations.

Conflict of interest

None declared.

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References

Abhyankar NS, Hall II EK and Hanagud SV (1993) Chaotic vibrations of beams: numerical solution of partial differential equations. *ASME Journal of Applied Mechanics* 60: 167–174.

Baruh H (1999) *Analytical Dynamics*. Boston: McGraw-Hill.

Debnath L (1997) *Nonlinear PDEs for Scientists and Engineers*. Boston: Birkhauser.

Doğan M (2006) Optimal nonlinear controller design for flexible robot manipulators. PhD Thesis. Istanbul: Boğaziçi University.

Doğan M and Istefanopulos Y (2007) Optimal nonlinear controller design for flexible robot manipulators with adaptive internal model. *IET Control Theory and Applications* 1: 770–778.

Doğan M and MorgülÖ (2010) On the control of two-link flexible robot arm with nonuniform cross section. *Journal of Vibration and Control* 16: 619–646.

Guo BZ (2002) Riesz basis property and exponential stability of controlled Euler–Bernoulli beam equations with variable coefficients. *SIAM Journal of Control and Optimization* 40: 1905–1923.

Hastings GG and Book WJ (1987) A linear dynamic model for flexible robotic manipulators. *IEEE Control System Magazine* 7: 61–64.

Kim JU and Renardy Y (1987) Boundary control of the Timoshenko beam. *SIAM Journal of Control and Optimization* 25: 1417–1429.

Krstic M, Siranosian AA and Smyshlyaev A (2006) Backstepping boundary controllers and observers for the slender Timoshenko beam: Part I – Design. In: *Proceedings of the American Control Conference*. Minneapolis, MN, pp. 2412–2417.

Krstic M, Guo BZ, Balogh A and Smyshlyaev A (2008) Control of a tip-force destabilized shear beam by observer-based boundary feedback. *SIAM Journal of Control and Optimization* 47: 553–574.

Luo ZH (1993) Direct strain feedback control of flexible robot arms: new theoretical and experimental results. *IEEE Transactions on Automatic Control* 38: 1610–1622.

Luo ZH, Guo BZ and MorgülÖ (1999) *Stability and Stabilization of Infinite Dimensional Systems with Applications*. London: Springer-Verlag.

Meirovitch L (1967) *Analytical Methods in Vibrations*. London: MacMillan.

Meirovitch L (2001) *Fundamentals of Vibrations*. New York: McGraw-Hill.

MorgülÖ (1991) Orientation and stabilization of a flexible beam attached to a rigid body: planar motion. *IEEE Transactions on Automatic Control* 36: 953–962.

MorgülÖ (1992) Dynamic boundary control of the Timoshenko beam. *Automatica* 28: 1255–1260.

Porter D and Stirling DSG (1990) *Integral Equations*. Cambridge: Cambridge University Press.

Reddy JN (1993) *An Introduction to the Finite Element Method*. New York: McGraw-Hill.

Sievers L, Balas MJ and Flotow AV (1988) Modelling of web conveyance systems for multi-variable control. *IEEE Transactions on Automatic Control* 33: 524–531.

Smyshlyaev A and Krstic M (2004) Closed-form boundary state feedbacks for a class of 1-D partial integro-differential equations. *IEEE Transactions on Automatic Control* 49: 2185–2202.

Smyshlyaev A and Krstic M (2005) Backstepping observers for a class of parabolic PDEs. *Systems and Control Letters* 54: 613–625.

Wang FY and Gao Y (2003) *Advanced Studies of Flexible Robotic Manipulators*. New Jersey: World Scientific.

Appendix A

Analytical solution of kernel

Note that the kernel expression given by (27) is obtained by changing variables to form a homogeneous hyperbolic kernel PDE in lieu of (22), with appropriate boundary conditions, as follows:

$$r(x,y) = k(x,y) + m\sinh(cx - cy) \tag{37}$$

$$r_{yy} - r_{xx} + fr = 0 \tag{38}$$

$$r(x,x) = -fx/2 \tag{39}$$

$$r(0,0) = 0 \tag{40}$$

$$r(x,0) = k(x,0) + m\sinh(cx) \tag{41}$$

$$r_y(x,0) = -c\cdot m\cosh(cx) \tag{42}$$

where $m = -\epsilon^{-1.5}/f$, and the terms produced by the second order spatial derivatives canceled each other. By performing integrals in (27) that contains Bessel functions $J_0(\cdot)$ and $J_1(\cdot)$, and after some lengthy calculations, we obtain the following explicit form for the kernel $k(x, y)$:

$$\begin{aligned} k(x,y) = {} & 0.5[h(0) + h(L)] + (c.m/4)(\sinh(cx) \\ & + \cosh(cx))\cdot\frac{1}{\sqrt{c^2+f}}\exp(-y\sqrt{c^2+f}) \\ & + \frac{1}{4}(\sinh(cx)+\cosh(cx))\cdot(m - 0.81039) \\ & \times[\exp(-y\sqrt{c^2+f}) - \exp(-cy)] \\ & - 0.62521\,x\,\frac{(1-\cos(y\sqrt{f}))}{2(m-0.81039)} + \frac{0.62521\,fy}{2(m-0.81039)} \\ & \times[1 - J_0(\sqrt{(y^2-x^2)f})] - m\sinh(cx-cy). \end{aligned} \tag{43}$$

Appendix B

Proof of asymptotic stability

After applying the control laws to the system (11)–(12), and after some lengthy but straightforward calculations, the time derivative of the Lyapunov function (28) is obtained as follows:

$$\begin{aligned} \dot{V} = {} & -2bc_o\dot{w}^2(L) + d_2\epsilon\|\dot{w}\|^2 + bd_2[-c_o\dot{w}(L)w(L) \\ & -\|w_x\|^2 - e\|w\|^2]. \end{aligned} \tag{44}$$

Note that Friedrichs' inequality ($\|\dot{w}\|^2 \le c_1\int_0^L \dot{w}_x^2\,dx + c_2\dot{w}^2(L)$) can be used to simplify the above equation with a suitable choice of c_2 and d_2. After this, we obtain

$$\begin{aligned} \dot{V} \le {} & -c_3\dot{w}^2(L) - c_4\int_0^L \dot{w}\dot{w}_{xx}\,dx - bd_2\|w_x\|^2 \\ & - bd_2e\|w\|^2, \end{aligned} \tag{45}$$

where the integral term, $c_4 < \dot{w}, \dot{w}_{xx} >$ is obtained by applying integration by parts to the term $c_1\int_0^L \dot{w}_x^2\,dx$. This inner product in $L_2(0, L)$ space, $< \dot{w}, \dot{w}_{xx} >$, can be rewritten by using equation (15) such that

$$< \dot{w}, \dot{w}_{xx} > = \frac{1}{b} < \dot{w}, \epsilon\,\dddot{w} > + e\|\dot{w}\|^2. \tag{46}$$

Note that the inner product can be bounded by Cauchy–Schwarz inequality:

$$\begin{aligned} & |< \dot{w}, \epsilon\,\dddot{w} >| \le \|\dot{w}\|\|\epsilon\,\dddot{w}\|, \\ & -\|\dot{w}\|\|\epsilon\,\dddot{w}\| \le < \dot{w}, \epsilon\,\dddot{w} > \le \|\dot{w}\|\|\epsilon\,\dddot{w}\|, \\ & < \dot{w}, \epsilon\,\dddot{w} > \ge -\|\dot{w}\|\|\epsilon\,\dddot{w}\|. \end{aligned}$$

After applying the time derivative and the triangle inequality to Equation (21), we get

$$\|\epsilon\,\dddot{w}\| \le \|\epsilon\,\dddot{u}\| + \|\int_0^x k(x,y)\,\epsilon\,\dddot{u}(y)dy\|.$$

Note that the kernel function is bounded in its domain—see Appendix A. By using the inequality in Porter (1990, Section 3.4), the following inequalities can be obtained for some positive constants K_3, $K_4 > 0$:

$$\|\epsilon\,\dddot{w}\| \le (1+K_3)\|\epsilon\,\dddot{u}\|, \tag{47}$$

$$< \dot{w}, \epsilon\,\dddot{w} > \ge -K_4\|\dot{w}\|\|\epsilon\,\dddot{u}\|. \tag{48}$$

We can use Equation (3) to get the following result:

$$< \dot{w}, \epsilon\,\dddot{w} > \ge -K_4\|\dot{w}\|\|b\,\dot{u}_{xx} - \dot{\alpha}_x - \epsilon\,x\dddot{\theta}_1\|. \tag{49}$$

Note that $\dot{\alpha}_x$ depends on $\|\dot{u}\|$ terms by Equation (10), and that the last term in the above inequality will decay exponentially due to decoupling. If ϵ times Equation (7) is subtracted from Equation (3), then we get

$$\epsilon\,b\,u_{xxxx} - \epsilon^2\,\ddot{u}_{xx} + bu_{xx} = \alpha_x.$$

This equation can easily be transformed into the well-known Klein–Gordon equation by replacing u_{xx} with y (Debnath, 1997). Besides, the right-hand side, namely α_x, will generate $\|\dot{u}\|$ terms as explained beforehand.

Thus it is proven that the $\dot{u}_{xx}$ term also depends on $\|\dot{u}\|$ terms for inequality (49). Finally, we have

$$< \dot{w}, \epsilon \dddot{w} > \ \geq -K_5 \|\dot{w}\| \|\dot{u}\|, \quad (50)$$

for some positive constant $K_5 > 0$. It can be shown with the same rationale as for inequality (47) that

$$\|\dot{w}\| \geq \ (1 - K_3)\|\dot{u}\|, \quad (51)$$

where $K_3 < 1$ can be set by scaling the kernel function. After multiplying both sides by $\|\dot{w}\|$, we get

$$< \dot{w}, \epsilon \dddot{w} > \ \geq -K_6 \|\dot{w}\|^2, \quad (52)$$

where $K_6 > 0$ is a positive constant. Equation (46) can be rewritten such that

$$< \dot{w}, \dot{w}_{xx} > \ \geq \left(-\frac{K_6}{b} + e\right)\|\dot{w}\|^2, \quad (53)$$

where b is a structural parameter and e is a controller parameter. By choosing the controller parameter e as $e > \frac{K_6}{b}$, we obtain $< \dot{w}, \dot{w}_{xx} > \ \geq \gamma \|\dot{w}\|^2$, where $\gamma = e - \frac{K_6}{b} > 0$. By using the above inequalities and (45), we obtain:

$$\dot{V} \leq -c_3 \dot{w}^2(L) - c_7 \|\dot{w}\|^2 - bd_2 \|w_x\|^2 - bd_2 e \|w\|^2, \quad (54)$$

$$\dot{V} \ \leq -K V, \quad (55)$$

where $K > 0$ is a positive constant. The asymptotic stability now follows from standard Lyapunov arguments. In fact, the decay is exponential.