Bell Soliton in Ultra-cold Atomic Fermi Gas

Ayan Khan

E-mail: ayankhan@fen.bilkent.edu.tr
Department of Physics, Bilkent University, 06800 Ankara, Turkey

Prasanta K. Panigrahi

E-mail: pprasanta@iiserkol.ac.in
Indian Institute of Science Education and Research-Kolkata, Mohanpur,
Nadia-741252, India

Abstract. We demonstrate the existence of supersonic bell soliton in the Bardeen-Cooper-Schrieffer-Bose-Einstein condensate (BCS-BEC) crossover regime. Starting from the extended Thomas-Fermi density functional theory of superfluid order parameter, a density transformation is used to map the hydrodynamic mean field equation to a Lienard type equation. As a result, bell solitons are obtained as exact solutions, which is further verified by the numerical solution of the dynamical equation. The stability of the soliton is established and its behavior in the entire crossover domain is obtained. It is found that, akin to the case of vortices, the bell solitons yield highest contrast in the BEC regime.

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1. Introduction

The experimental observation [1] of superfluidity in the Bardeen-Cooper-Schrieffer-Bose-Einstein condensate (BCS-BEC) crossover regime has given impetus to the study of ultra-cold trapped Fermi gases at unitarity [2, 3]. Though the beginning of the experimental study of superfluidity in cold atomic gases dates back to the observation of BEC, a decade and half ago [4, 5], the interaction controlled superfluidity, where atomic Fermi gas changes constituents from Cooper pairs to composite bosons, is fairly new [6]. As the atoms pass through a situation, where the inter-atomic s-wave scattering length diverges (the unitarity), several novel phenomena have been observed, some having strong resemblance with characteristics of high-T_c superconductivity [7]. The theory of crossover (the interaction driven study from BCS to BEC passing the unitarity) was first suggested long time back [8] and has been studied systematically in the early eighties [9, 10]. The recent advancement in experiments [11, 12, 13, 14, 15] has opened up avenues for deeper understanding of nonperturbative aspects of this strongly interacting system, in a controlled manner.

Lower dimensional nonlinear systems are known to exhibit counter intuitive features, which are not possible in the three dimensional world. A classic example being existence of bright and dark solitons in quasi one dimensional BEC [16, 17, 18, 19, 20, 21, 22, 23, 24, 25], which derive their stability from the balancing of nonlinear effect with dispersion. The spreading effect of dispersion is exactly compensated by non-linearity, resulting in these stable structures [26]. Here, we show the existence of bell shaped solitons (bell-antibell pair) in the ultra-cold atomic gases which are non-topological in nature. We note that a soliton solution can be classified in many ways, depending upon its shape (bell, kink, breather); nature of the non-linear systems (thus governing non-linear equations) which lead KdV (pulse) soliton, sine-Gordon soliton, non-linear Schrödinger (NLS) soliton (envelop soliton); depending on its topology such as topological and non topological solitons etc. Contrary to the envelop (bright) solitons, a Bell soliton is generally obtained from a KdV type of equation and the amplitude of the soliton is directly proportional to its velocity [27]. Similar hyperbolic secant pulse profiles have been observed in a number of natural systems, the most notable being the phenomenon of self-induced transparency in atomic media [28]. We establish the conditions under which such localized structures can manifest in the fermionic cold gases. This is explicated through exact solution in a controlled mean field approach, which enables us to observe the pulse dynamics, starting from the fermionic side and ending up in the Bose-Einstein condensate.

In the context of the BCS-BEC crossover, mean-field microscopic description is well captured by the Bogoliubov-de Gennes (BdG) equation [29]. The analysis of BdG equation is considerably more involved as compared to the Gross-Pitaevskii (GP) equation, describing the mean field dynamics of BEC. Antezza *et al.*, have numerically solved the BdG equation in a box and discussed the behavior of dark soliton, in a superfluid Fermi gas along the entire crossover domain, in a quasi-one dimensional (Q1D) geometry [30]. Similar approach was also adopted in the later studies as well [31, 32]. Manini *et al.*, have developed a simpler approach, using extended Thomas-Fermi density functional theory (ETFDFT), which incorporates the interaction through polytropic density of state [33], where it was shown that for ^6Li, it is possible to accurately find the collective properties in the crossover regime. More recently, Wen *et al.*, have presented the dark and bright soliton solutions through the route of perturbative analysis of the above mean field equation [34, 35]. This leads to Korteweg-de Vries (KdV) equation whose soliton solutions are well known.

Before going into the details of our analysis, here we like spend some time on the pros and cons of the ETFDFT model. As mentioned before, a BdG analysis is one of the most robust way of treating the unitary Fermi gas as it takes into account the main contribution to the kinetic energy, and treat it exactly in noninteracting systems even with a nonuniform density spatial variation. Where as this approach gives the exact kinetic energy only for a uniform system and even when extended with the addition of gradient and higher order derivatives of the density, the ETF functional is not able to reproduce shell effects in the density profile. But in this model one need not to worry about the number of particles which is a big concern in BdG approach. More over through ETFDFT one can obtain the qualitative physical picture at zero temperature with relative numerical ease. Thus it is worth looking at the unitary Fermi system using this model.

Here, we present (i) an exact bell soliton solution from the ETFDFT by showing a direct connection of ETFDFT with Lienard type of equations. Thereby we obtain the soliton solution which does not necessitate any perturbative approach [34, 35]. These solitons have highest contrast in the BEC regime, similar to the earlier result for vortices and dark solitons [30, 36]. (ii) We explore the collective excitations and obtain the sound velocity which shows an explicit dependence on the phenomenological λ parameter (unless we absorb the quantum pressure term inside the chemical potential). (iii) The stability of these solitons are discussed in terms of Noether charge. In the last few passages we present (iv) a scheme to treat ETFDFT analytically when the ultracold atomic Fermi gas is subjected to a harmonic trap and we observe the resulting oscillating dynamics of the soliton due to the parabolic confinement.

The article is organized as follows: In Sec. 2, we discuss the polytropic approximation for obtaining the relevant equation of state in the crossover regime. A brief account is also presented on dimensional reduction of mean-field dynamics, to deal with quasi 1D scenario. In Sec. 3, we have presented the exact connection between the ETFDFT and Lienard type equation, from which we obtain bell soliton solutions. We then explore its evolution from weak to strong coupling regimes. Further, we present the oscillating dynamics of the soliton and establish its supersonic nature by analyzing the sound velocity in this medium. The stability is analyzed in the light of Noether charge and associated energy. Sec. 4 is devoted for the concluding remarks and future directions of study.

2. Polytropic Approximation and Quasi-One-Dimensional Dynamics

A. Interaction in Unitary Fermi Gas

In recent years, Manini *et al.*, have shown a simple way to investigate the unitary Fermi gas without losing its essential essence [33]. In particular, for ^6Li they have built an efficient parametrization of the energy per particle, based on Monte Carlo data and asymptotic behavior of the interaction. Prior to that, it was shown that a power law dependence of equation of state can be effective to find reliable expressions for the collective frequencies which is analytically treatable but preserves the vital physical picture [37]. Later it was demonstrated that, at zero temperature under local density approximation the homogeneous chemical potential can be written in a series expansion of gas parameters with good effect [38].

Here, we make use of this ETFDFT approach to investigate the existence of solitonic modes in superfluid Fermi gas. We start from the ETFDFT equation, which controls the dynamics of the trapped superfluid Fermi gas [39]:

$$i\hbar\frac{\partial}{\partial t}\Psi = [-\frac{\hbar^2}{4m}\overrightarrow{\nabla}^2 + 2U(\mathbf{r}) + 2g(|\Psi|^2) + (1-4\lambda)\frac{\hbar^2}{4m}\frac{\overrightarrow{\nabla}^2|\Psi|}{|\Psi|}]\Psi, \tag{1}$$

where Ψ is the Ginzburg-Landau order parameter at zero temperature and the normalization condition $\int d\mathbf{r}|\Psi|^2 = N$ (N being the total atomic pair number of the superfluid Fermi gas) and $U(\mathbf{r})$ is the trapping potential. The bulk chemical potential or equation of state is characterized by $g(n)$. We can introduce the polytropic approximation for equation of state as $g(n) = \mu_0 n^\gamma$. As discussed earlier, the polytropic approximation holds good at zero temperature under local density approximation which is essentially a perturbative approach capable of providing deep insight into the physical nature of the quantum system [38]. Under the polytropic approximation, μ_0 is the reference chemical potential, which we will explain after a short while, and n is the atomic density. It is worth noting that, the interaction term is always positive in the whole BCS-BEC crossover. We consider λ as a phenomenological parameter, accounting for the increase of kinetic energy due to the spatial variation of the density.

The motivation to introduce the λ comes from the density functional theory where a gradient correction was introduced to account for the non uniformity of the system [40]. An energy functional for fermions in terms of the density and its derivatives is usually called extended Thomas-Fermi (ETF) functional. Now if we define $n(\mathbf{r})$ as the local number density then a quantity like $-\lambda\hbar^2\nabla^2\sqrt{n}/(2m\sqrt{n})$ can be interpreted as a next to leading term correction factor, where m being the atomic mass. There exits some ambiguity regarding the value of λ as well. Initially the value was considered as 1 in context of BCS-BEC crossover [33] but later it was suggested that $\lambda = 1/4$ is better choice based on its good agreement with the theoretical epsilon analysis around $d = 4-\epsilon$ spatial dimensions in unitary regime [40]. For $\lambda = 1/4$, Eq.(1) reduces to a form similar to the GP equation. Since the formation of soliton is due to the delicate balance between dispersion and nonlinearity, and in quasi one dimensional (Q1D) systems, the quantum pressure has significant contribution to dispersion, therefore one must pay careful attention to λ. Further, in Q1D the particle number of the condensate is small, which makes the contribution of quantum pressure more significant. For large number of atoms, the radial contribution becomes significant. The ground state wave function in the radial direction is Thomas-Fermi (TF) type, which results in discrete eigenmodes of linear excitations in the radial direction, and hence induces a dispersion effect on the wave propagation in axial direction. The resulting dispersion can balance the nonlinear interaction and make the system more favorable to form solitons.

The essence of the interaction driven crossover is embedded in the polytropic coefficient γ. Therefore, before going into the details of calculation, we briefly summarize the behavior of γ and associated reference chemical potential μ_0. Here, the weak coupling to strong coupling evolution is captured through the following relation [33],

$$\gamma = \frac{\frac{2}{3}\epsilon(y) - \frac{2y}{5}\frac{\partial\epsilon(y)}{\partial y} + \frac{y^2}{15}\frac{\partial^2\epsilon(y)}{\partial y^2}}{\epsilon(y) - \frac{y}{5}\frac{\partial\epsilon(y)}{\partial y}}, \tag{2}$$

where $\epsilon(y)$ is an analytical function, whose parameters are fixed using the Monte Carlo data [41], and $y = (k_F a)^{-1}$. If one plots Eq. (2) one will find that $\gamma \in [2/3, 1]$ (roughly), while $y \in [-1, 1]$, thus at $y = 1$ Eq. (1) effectively reduces to the Gross-Pitaevskii equation. Where as, in the Fermi system the ground state energy is proportional to (density)$^{2/3}$ therefore, one can obtain the BCS like picture while $\gamma \simeq 2/3$ and thereby it becomes possible to retain the essence of both the limits. In a similar fashion, the reference chemical potential can be defined as,

$$\mu_0 = \epsilon_F \left[\epsilon(y) - \frac{y}{5}\frac{\partial\epsilon(y)}{\partial y}\right]. \tag{3}$$

The study of the superfluids at zero temperature can take various routes. One simple yet effective way is to study the hydrodynamics of the system. The ground state solution of superfluid hydrodynamic equations corresponds to $\partial n_s/\partial t = 0$ and $\mathbf{v}_s = 0$ (n_s, $\mathbf{v}_s$ being the superfluid density and velocity components)[35]. As a result, it is possible to determine the ground state chemical potential (μ_g) of the superfluid system: $\mu_0(n_q(\mathbf{r})) + U(\mathbf{r}) = \mu_g$. Here n_q denotes equilibrium density, which can be related to the reference density through the polytropic approximation as $n_q = n_0[\frac{\mu_g - U(\mathbf{r})}{2\mu_0}]^{1/\gamma}$, n_0 is particle number density. One can obtain μ_0 from Eq.(3) and n_0 is taken as the density of the ideal Fermi gas at the trapping center of the system. The external potential is the usual harmonic trap, which in cylindrical coordinate takes the form, $U_{x,y,z} = \frac{1}{2}m(\omega_\perp r^2 + \omega_z^2 z^2)$, with $r = \sqrt{x^2 + y^2}$. The TF radius in radial and axial directions can be written as, $R_{xy} = 2\mu_g/m\omega_\perp^2$ and $R_z = 2\mu_g/m\omega_z^2$ respectively. Applying the normalization condition for equilibrium density ($\int n_q d\mathbf{r} = N$) the ground state chemical potential can be written as [35],

$$\mu_g = 2\epsilon_F \left[\left\{\epsilon(y) - \frac{y}{5}\frac{\partial\epsilon(y)}{\partial y}\right\}^{1/\gamma} \times \frac{\sqrt{\pi}(1+\gamma)\Gamma(\frac{1}{\gamma}+\frac{5}{2})}{8\gamma\Gamma(\frac{1}{\gamma}+2)}\right]^{2\gamma/(2+3\gamma)},$$

where $n_0 = (2m\epsilon_F)^{3/2}/6\pi^2\hbar^3$ and $\epsilon_F = \hbar(6N\omega_\perp^2\omega_z)$. In the preceding sections, it will be established that μ_g plays a crucial role in shaping the dynamics of soliton and also sound propagation in this medium.

B. Dynamics in Quasi One Dimension

To model a cigar shaped trap, it is essential to consider that the interaction energy in the transverse direction is much less than the kinetic energy [42]. The fermions are assumed to be confined in a cylindrical harmonic trap, with radial frequency much larger than the axial one, i.e., $\omega_\perp >> \omega_z$. The important assumption we need to make for employing the Gaussian variational ansatz in BEC in such situations is that, the interaction energy per particle must be weaker than the zero point energies associated with the harmonic confinement [43]. Though this method has its own limitation, it is widely used in dilute Bose gas. Since the current ETFDFT model also lies on dilute limit where the inter-particle separation is much smaller than the s-wave scattering length therefore we are safe to use the similar Gaussian ansatz here as well.

To reduce Eq.(1) into Q1D form by using Gaussian ansatz, we consider

$$\Psi(\mathbf{r},t) = \frac{\sqrt{N}}{\sqrt{\pi a_\perp^2\sqrt{\lambda}}}\Phi\left(\frac{z}{a_\perp},\omega_\perp t\right) \times \exp\left(-i\sqrt{\lambda}\,\omega_\perp t - \frac{x^2+y^2}{2a_\perp^2\sqrt{\lambda}}\right), \tag{4}$$

where $a_\perp = \sqrt{\hbar/m\omega_\perp}$. The reduced nonlinear Schrödinger equation (NLSE) then becomes,

$$i\frac{\partial\Phi}{\partial t} = -\frac{1}{4}\frac{\partial^2\Phi}{\partial z^2} + \omega_0 z^2\Phi + 2\mu_0(|\Phi|^2)\Phi + (1-4\lambda)\frac{1}{4}\frac{\frac{\partial^2|\Phi|}{\partial z^2}}{|\Phi|}\Phi, \tag{5}$$

where $\omega_0 = \omega_z/\omega_\perp$, and μ_0 is appropriately normalized by $\hbar\omega_\perp$.

We first analyze the results without the trap, and later we will incorporate the trap to obtain analytic solutions. The solution of the above NLSE, without the pressure and trap have been studied in different perspective [44, 45]. Using a traveling wave ansatz of the type $\Phi(z,t) = \rho(z,t)\exp i(\chi(z,t) + \mu_g t)$, we study the system in the comoving frame, $\zeta = z - Vt$, where V is the soliton velocity. After applying these transformation in Eq.(5), the imaginary equation (imaginary part of Eq.(5)) arising from the current conservation gives,

$$\chi_\zeta = \frac{4C_0}{\rho^2} + 2V, \tag{6}$$

where C_0 is an integration constant. We consider the phase and amplitude as uncorrelated and hence assume $C_0 = 0$. The real equation (real part of Eq.(5)) then reads,

$$\lambda\rho_{\zeta\zeta} + (V^2 - \mu_g)\rho - 2\mu_0\rho^{2\gamma+1} = 0, \tag{7}$$

where we have used the results from Eq.(6). The subscript ζ means derivative with respect to ζ. It is worth noting that for $\gamma = 1$, one can have full set of Jacobi elliptic function as solutions. For Fermi gas, we do not have such freedom and we need to solve the equation, considering γ as unknown. Later on we will use selected values of γ obtained from Eq. (2).

3. Localized Structure in the Crossover Regime

A. Bell Soliton

Till date only a few studies have addressed the complexity involving the polytropic assumption. Very recently, Wen *et al.* [34, 35], made such efforts and studied both dark and bright solitons using a perturbative expansion scheme on the mean field equation. Interestingly, there exists a transformation which will enable us to reduce this equation to a well known form, which has exact solutions. Without loss of generality one can take $\rho = \sigma^{1/\gamma}$, which connects Eq. (7) to a Lienard type equation:

$$\frac{1}{\gamma}\sigma\sigma_{\zeta\zeta} - \frac{\gamma-1}{\gamma^2}\sigma_\zeta^2 + P_1\sigma^2 - P_2\sigma^4 = 0, \tag{8}$$

where $P_1 = (V^2 - \mu_g)/\lambda$ and $P_2 = 2\mu_0/\lambda$. We like to mention here that one can obtain a real constant solution from Eq.(8) iff $V^2 - \mu_g > 0$, as $\sigma = \sqrt{P_1/P_2}$. But soon we will see that the prevailing condition is inconsistent while we try to obtain a localized solution. Thus we refrain ourselves from the constant solution. To obtain a localized solution we apply fractional transformation [45] of the following form,

$$\sigma = \frac{A\,\mathrm{sech}^2[B\zeta/2]}{1 + C\,\mathrm{sech}^2[B\zeta/2]}. \tag{9}$$

As short recap, we like to remind the readers that fractional transformation is an useful tool to obtain solutions from a non-linear equation. In this approach, one can consider an ansatz solution in the following way,

$$\sigma(\zeta) = \frac{A + Bf^\delta(\zeta)}{C + Df^\delta(\zeta)},$$

where $AD - BC \neq 0$, δ can be any power of the function f. A, B, C, D are constant which can be evaluated through consistency conditions after applying this ansatz in the main equation and collecting the coefficients according to the power of f. A comprehensive description of this process can be found in Ref. [46].

We follow the same prescription, thereby application of Eq.(9) in Eq.(8) leads to the following set of consistency conditions:

$$\begin{aligned}
-16\gamma P_1 + 16B^2\gamma - 32B^2 &= 0,\\
-2C\gamma P_1 - 16B^2\gamma + B^2C + 40B^2 &= 0,\\
\text{and } A^2\gamma P_2 - C^2\gamma P_1 - 8B^2C &= 0.
\end{aligned}$$

The obtained solutions are,

$$A = \pm\frac{1}{2}\sqrt{\frac{\gamma P_1 - 3P_1}{\gamma P_2 - 2P_2}},\ B = \pm\sqrt{\frac{\gamma P_1}{\gamma - 2}} \text{ and } C = -1/2.$$

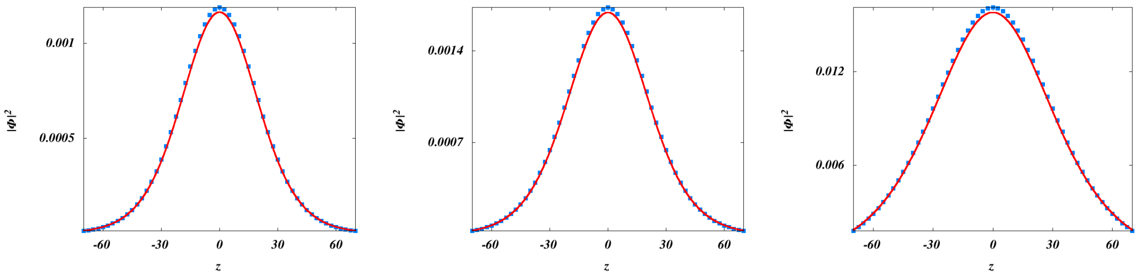

Figure 1. (Color online)The bell soliton profiles across the BCS-BEC crossover regime is depicted. The interaction is tuned from weak to strong ($1/k_Fa = -1$, 0, 1) from top to bottom. The red line is our analytic result and the blue squares denote corresponding numerical solutions after evolution of the Q1D ETFDFT equation.

Since our motivation is to find localized solutions, it is important to impose $\mu_g - V^2 > 0$ (contrary to the constant solution), otherwise the solution will be of trigonometric form (since $\gamma - 2 < 0$ always for this kind of system) having singularities in time evolution. One should note that, the above assumption actually imposes a ceiling on the soliton velocity contrary to the usual bright soliton solution whose velocity is not bounded. Incorporating this assumption, the solution can be simplified to the compact form (after normalization),

$$\rho(\zeta) = \left\{\frac{\eta e^{i\pi/2}}{2\sqrt{B(-1,\frac{1}{\gamma},\frac{\gamma-2}{\gamma})}}\sqrt[4]{\frac{\gamma(V^2-\mu_g)}{\lambda(\gamma-2)}}\right\}^{1/\gamma} \times \left\{\text{sech}\left[\sqrt{\frac{\gamma(V^2-\mu_g)}{\lambda(\gamma-2)}}\zeta\right]\right\}^{1/\gamma} \quad (10)$$

where $\eta = \pm 1$ is the polarity of the soliton (bell/anti-bell) and $B(-1,\frac{1}{\gamma},\frac{\gamma-2}{\gamma})$ is the incomplete beta function. It is worth pointing out that, this is a distinct and new solution (bell type), as compared to the solution presented in Ref. [35]. Fig.(1) demonstrates the bell soliton at time $t = 0$, for different couplings, with $V^2 = 0.9\mu_g$. The solid red line is our analytic result (Eq. (10)), the corresponding numerical solution is depicted through blue squares. The numerical result presented here are obtained via solving ETFDFT equation using the split-step Crank-Nicolson method by discretizing in space and time [47]. The discretized equation is then solved by propagation in imaginary time over small time steps (0.00002, total time step is taken as 8000). Both the results agree appreciably well with each other.

One must notice that in Eq.(10), if $\gamma = 2$, the solution will be unstable as referred in Ref. [44] but in our case on physical grounds $\gamma < 2$ (though mathematically γ can have any value but in this model γ values are fixed by Eq.(2), thus its value can never exceed 1). It is important to observe the change in visibility of the bell solitons across the crossover. The visibility/amplitude of the bell soliton is different in different superfluid regimes. For dark soliton, it has already been shown that visibility is maximum in the BEC side [30]. In Fig.(1), we notice that the visibility does increase as we move from weak coupling to strong coupling regime. Further, one can note that the change in visibility is more rapid in BEC side and this increase is monotonic. A qualitative comparison with the visibility of vortex core density in the BCS-BEC crossover [36] reveals striking resemblance with our study.

B. Sound Velocity and Stability Criterion

Before presenting the sound velocity result in our system, we intend to address one intriguing issue regarding the sound velocity. It is well know that one can calculate the sound velocity through the hydrodynamics as well as from the ETFDFT equation (Eq.1)[39, 43]. It is worth noting that the effect of quantum pressure does not enter in the sound velocity expression if we use the hydrodynamic approach ($c_s = \sqrt{\gamma\mu_g}$). This is even true for ETFDFT approach if we absorb the quantum pressure term in the chemical potential. But otherwise the quantum pressure term will leave an effect on the sound velocity.

We now analyze the soliton propagation velocity, for the bell solitons and relate it to the sound velocity. A general way to obtain the sound velocity is by applying a small perturbation about a constant background. A systematic calculation reveals, $c_s = \sqrt{2\lambda\mu_0\gamma}\rho_0^\gamma = \sqrt{2\lambda\gamma\mu_g}$ (since $\rho_0^{2\gamma} = \mu_g/\mu_0$) considering the homogeneous unperturbed condensate density varies slowly in space. One should note here that the contribution arising from the quantum pressure (λ) also makes a presence in determining the sound velocity. Qualitatively our results are in good agreement with Ref. [35] but small quantitative difference might be due the fact that, our analysis is very different in nature from the previous analysis where collective excitations were calculated from the density fluctuation in three dimension. As an example, in the deep BEC limit with $\gamma = 1$ and $\lambda = 1/4$, $c_s^{BEC} = \sqrt{\mu_g/2}$ as obtained in Ref. [35]. We have seen that condition for existence of bell soliton is $V^2 < \mu_g$. From the above discussions on sound, one can write $V^2 < c_s^2/(2\lambda\gamma)$, which allows a wide range of soliton velocity. However, this model accepts a certain range of values for γ. More precisely, $\gamma \in [2/3, 1]$. Hence it turns out that $V \in [1.73c_s, 1.41c_s]$, which points to the supersonic nature of the soliton velocity.

Table 1. Comparison of different parameters from weak to strong coupling limit for $V^2 = 0.9\mu_g$.

$\frac{1}{k_F a_s}$	γ	μ_0	μ_g	$c_s\|_{\lambda=0.25}$	$c_s\|_{\lambda=0.13}$
-1	0.624	0.758	1.92	0.773	0.558
0	0.666	0.420	1.65	0.741	0.534
1	1.053	0.083	0.59	0.557	0.401

To understand the origin of bell soliton generation, we must look back at the role of the ground state chemical potential. Generaly dispersion relation can be written as, $\omega \propto k$ where ω and k is the frequency and wave number (momentum) of the system respectively. This is usually referred as material dispersion. But there exists another dispersion mechanism which is known as waveguide dispersion. This dispersion actually contains the dimensional effect i.e $\sqrt{\omega^2 - \omega_d^2} \propto k$ where ω_d depends on the dimension of

Table 2. Comparison of conserved charge and associated energy of the bell solitons in the BCS-BEC crossover regime for different soliton velocity ($\lambda = 1/4$).

$\frac{1}{k_F a_s}$	$V^2 = 0.75\mu_g$		$V^2 = 0.8\mu_g$		$V^2 = 0.85\mu_g$		$V^2 = 0.9\mu_g$		$V^2 = 0.95\mu_g$	
	$\mathcal{Q}\sqrt{\mu_g}$	$\mathcal{E}(\mathcal{Q})$	$\mathcal{Q}\sqrt{\mu_g}$	$\mathcal{E}(\mathcal{Q})$	$\mathcal{Q}\sqrt{\mu_g}$	$\mathcal{E}(\mathcal{Q})$	$\mathcal{Q}\sqrt{\mu_g}$	$\mathcal{E}(\mathcal{Q})$	$\mathcal{Q}\sqrt{\mu_g}$	$\mathcal{E}(\mathcal{Q})$
-1	23.893	29.8898	18.682	20.561	13.604	12.762	8.7001	6.577	4.051	2.164
0	40.154	43.6536	32.112	30.137	24.074	18.765	16.039	9.691	8.011	3.186
1	25.002	35.8255	22.615	25.688	19.870	16.730	16.559	9.142	12.124	3.254

the system. In our system, though the trap is effectively cigar shaped, we have taken into account the radial contribution in the density through μ_g. Thus the TF distribution in the ground state, incorporated through μ_g, supports the "waveguide dispersion", as opposed to the normal "material dispersion", and we consider this different dispersion mechanism supports the generation of bell solitons [35].

For completeness, we list the values of several relevant parameters explicitly for different couplings in Table 1. All the parameters are already normalized by the Fermi energy or Fermi velocity according to the relevance. The gradual decrease of reference chemical potential, as well as ground state chemical potential from BCS to BEC regime is understandable as it is known that the BCS state is highly dense/closely packed, whereas the composite bosons are sparsely distributed. The gradual decrease of sound velocity, as one moves towards strong coupling regime also agrees with the experimental data [48]. The quantitative analysis of our result reveals that, the most suitable value for the phenomenological parameter turns out to be $\lambda = 0.13$ (which has been proposed in Ref. [39]). We also note that the calculated sound velocity using $\lambda = 0.13$, augurs well with the sound velocity obtained through mean-field BCS-BEC crossover formalism [49].

To investigate the stability of these bell solitons, which are non-topological in nature, one should calculate the Noether charge defined as $\mathcal{Q} = \int |\rho|^2 dz$, and the corresponding energy,

$$\mathcal{E}(\mathcal{Q}) = \int \left[\lambda \Big|\frac{\partial \rho}{\partial z}\Big|^2 + \mu_0(|\rho|^2)^{2\gamma}\right] dz.$$

The stability of these solitons demands that $\mathcal{E}(\mathcal{Q}) < \sqrt{\mu_g}\mathcal{Q}$ [50, 51]. After evaluating the charge and energy integrals numerically, we list our result in Table 2. Using these results and the criterion for stability we find that solitons are more stable in the high velocity regime (in the vicinity of $V \simeq \sqrt{\mu_g}$). At relatively lower velocity ($V \simeq \sqrt{0.75\mu_g}$) the whole crossover is unstable but gradual increase in the velocity stabilizes the system for different interaction regime. At $V = \sqrt{0.8\mu_g}$ interestingly one can see that, the crossover regime is stable where as, the two extreme regimes are unstable. Afterwards, ($V \simeq \sqrt{0.85\mu_g}$), the entire crossover satisfy the stability criterion. Hence, the onset of stability happens faster in the unitary regime, which is sensible as different experiments already pointed out that crossover is more stable compared to BCS and BEC sides [1].

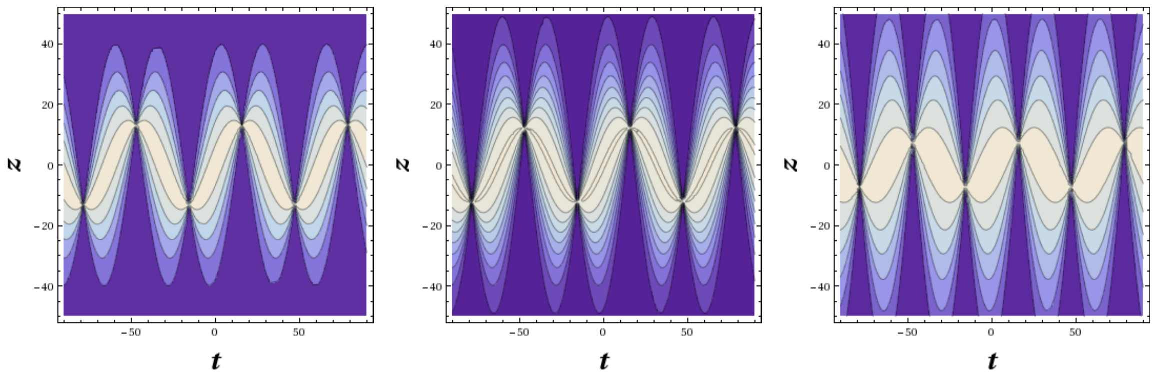

Figure 2. (Color online) Time evolution of of the bell soliton shows oscillation inside the trap. The interaction is tuned from weak to strong ($1/k_F a = -1, 0, 1$) depicted from left to right in the figure.

C. Coherent Controll of Bell Solitons

We will now extend our analysis in presence of harmonic confinement. For that purpose we start from Eq. (5) with interaction coefficient being time dependent i.e $\mu_0 \equiv \mu(t)$ as suggested in Ref.[52]. Since we are interested in finding soliton solutions, a self similar ansatz of the following form can be employed [52, 53],

$$\Phi(z,t) = \sqrt{\alpha(t)}\rho[\alpha(t)(z - l(t))] \times \exp\left[i(\theta(z,t) + \chi\{\alpha(t)(z - l(t))\})\right], \quad (11)$$

where the phase has the exact form, $\theta(z,t) = \mu_g \int_0^t \alpha(t')^2 dt' - c(t)z^2$, and χ being the density dependent complex phase of the solution profile. Application of the ansatz in the NLSE and separation of real and imaginary parts result in two equations, related to the continuity (imaginary part) and pressure (real part) equations.

The following consistency conditions emerge from Eq. (5),

$$\frac{d\alpha(t)}{dt} = \alpha(t)c(t), \qquad c(t)l(t) + \frac{dl(t)}{dt} = V, \quad (12)$$

The second equation in Eq. (12) represents conservation of the velocity components, where V being a conserved quantity having the dimension of velocity.

The equation of continuity leads to the following phase relation,

$$\chi_T = \frac{4C_0}{\rho^2} + 2V, \quad (13)$$

where $T = \alpha(t)(z - l(t))$. As before, to avoid the amplitude dependence on the phase, we consider the integration constant $C_0 = 0$.

It is interesting to see that $c(t)$ satisfies a Riccati type equation, $\frac{dc}{dt} - c^2 = \omega_0^2$, which can be mapped to Schrödinger equation, through a transformation: $c(t) = -\frac{d \ln \beta(t)}{dt}$. This relation enables us to find the soliton profile for different type temporal variations of the trap frequency ω_0 [54]. At present, we will only demonstrate the effect for usual harmonic potential for which ω_0 is constant. This leads to the solution $c(t) = \omega_0 \tan[\omega_0 t]$ and $\alpha(t) = \alpha_0 \sec[\omega_0 t]$. At this point, we can shed further light on the role of V. The presence of the trap necessitates chirping of the phase, which enforces another velocity component. In the absence of the trap, it is appropriate to consider $c(t) = 0$, and then we can easily get back the well known velocity expression as, $l(t) = Vt$.

The amplitude equation in the comoving frame leads to,

$$\lambda\rho_{TT} + (V^2 - \mu_g)\rho - 2\tilde{\mu}(\rho^2)\rho = 0. \tag{14}$$

The third term, involving the nonlinear interaction can now be written more precisely using polytropic approximation: $2\tilde{\mu}(\rho^2)\rho = \mu_0\alpha\rho^{2\gamma+1}$, where $\tilde{\mu} = \frac{\mu(t)}{\alpha^2}$ and $\mu(t) = \mu_0\alpha^{3-\gamma}$. Thus the temporal variation of the interaction is coupled with width of the solution, as was observed in in Ref. [52]. It must be noted that Eq.(14) is same as Eq.(7). Hence one can find the same solution with distributed coefficients (in the homogeneous case, $\alpha(0) = 1$).

Fig.(2) depicts the soliton oscillation at the entire crossover regime. One can find similar type of solution in Ref. [53, 55, 56] only for the BEC regime. Oscillation of soliton inside a trap in BEC is already a well documented fact [52, 53, 57, 58, 59, 60]. It has been concluded that the oscillation frequency of the soliton is $\frac{\omega_z}{\sqrt{2}}$. Recent study of dark soliton in the Fermi gas has predicted that the oscillation in the unitary regime will be $\frac{\omega_z}{\sqrt{3}}$ [61, 62]. In our analysis we consider $\omega_0 = \omega_z/\omega_\perp = 0.1$, which is reasonable if we keep in mind the actual experimental situations. It should be mentioned that the oscillation frequency does not show any dependence on the inter atomic interaction, which might be due to the limitation of the hydrodynamic approach.

4. Conclusion

In precise, here we have demonstrated localized solitons in the Fermi gas of ^6Li by exploiting an analytical connection between ETFDFT and Lienard equation. These solutions in quasi 1-D system have also been checked numerically, using split-step Crank-Nicolson method. The solitary waves draw support from the radial contribution incorporated through μ_g. The analysis of elementary excitations reveals the explicit dependence of sound velocity on the phenomenological parameter λ, unless we absorb the quantum pressure inside the chemical potential. The propagation velocity of the bell soliton is supersonic and there exists a fixed upper bound of the velocity beyond which the system will be unstable. The upper bound is controlled by the ground state chemical potential in the equilibrium condition. We realize that the stability of solitons across the crossover is bounded in a narrow region of velocity, approximately, $0.8\sqrt{\mu_g} < V < \sqrt{\mu_g}$. The lower bound is interaction dependent. It reveals that the stability is attained at unitarity at lower velocity than the other two regions. This matches well with the recent finding that, the superfluid Fermi gas is more stable at unitarity [1]. In the last part, we have shown the mathematical technique to tackle inhomogeneous ETFDFT analytically.

In this article we have described how to obtain certain kind of localized solution in ultra cold Fermi gas using ETFDFT technique. Though this model has its own limitations (as mentioned before) but it allows you to make good qualitative idea about the strongly correlated Fermi system. As a matter of fact, always one trades between simplicity and accuracy. Thus we consider evolution of bell soliton through real space BdG analysis using the polytropic density of state as an exciting next step in the near future.

As a concluding remark, we hope that these solitons can be experimentally verified in near future. In experiments, solitons can be created in a similar way as vortices are sometimes made [1]. A bell soliton can be created on the BEC side of the crossover and then the Feshbach resonance can be tuned towards unitarity or the BCS side. Although vortices are made by imposing rotation on a BEC, bell solitons are made via density engineering i.e a sudden switch on of a focused and blue-shifted laser beam should initiate a density flush thus producing bell soliton. Current rapid growth as well as interest in ultracold atom research (specially in Fermi systems) encourage us to believe that these solitons will be visible in the experiments soon.

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