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J. Stat. Mech. (2014) P10014

(http://iopscience.iop.org/1742-5468/2014/10/P10014)

Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

A disagreement-percolation type uniqueness condition for Gibbs states in models with long-range interactions

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Received 28 July 2014
Accepted for publication 23 August 2014
Published 9 October 2014

Online at stacks.iop.org/JSTAT/2014/P10014
doi:10.1088/1742-5468/2014/10/P10014

Abstract. We extend a condition for the uniqueness of Gibbs states in terms of percolation in the coupling of two independent realizations to lattice spin models with long-range interactions and a not necessarily unique ground state.

Keywords: rigorous results in statistical mechanics, classical phase transitions (theory), percolation problems (theory)

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1742-5468/14/P10014+07$33.00

Contents

1. Introduction

There are several different approaches to the problem of the uniqueness of Gibbs states. The most vigorous and comprehensive is the method of cluster expansions, where the local characteristics of a field are depicted in terms of clusters. Unfortunately, this method needs a small parameter and works only in the low-temperature ferromagnetic or high temperature regions, in the region of large magnetization, etc. The concept of cluster expansions goes back to [1]. Later on, the rigorous mathematical theory of cluster expansions was developed in [2–11]. The other known theory called the 'Dubrushin uniqueness method', which is based on a kind of contraction argument and which is the method of estimating the overall interaction of a spin with all other spins [12]. There is also a unique condition applicable only to 1D models, which requires the finiteness of the total interaction energy of the spin on any two complementary half-lines [13–17]. An alternative method for establishing the absence of phase transition reduces the uniqueness problem to that of the percolation of special clusters [18]. This method is especially powerful in 1D models with very slowly decreasing potentials when the classical methods mentioned above fail to work. The origin of the main idea of this method goes back to [19], where the uniqueness theorem for 1D antiferromagnetic models was established. An apparent deficiency of this method is the fact that it works only in models with a unique ground state. In the present paper we generalize the method by eliminating the condition of ground state uniqueness.

2. The main result

Let G be connected, a countable infinite and locally finite graph. We assume that the spin variables $\phi(x) \in \Phi$, where Φ is a finite set. Let:

$$H_0(\phi) = \sum_{<x,y>} J(\phi(x), \phi(y))$$

where the summation is taken over all the nearest neighbors $\langle x, y\rangle; x, y \in G$, and $J(\phi(x), \phi(y))$ is a translationally invariant function.

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Consider a model on G with the formal Hamiltonian:

$$H(\phi) = H_0(\phi) + \sum_{B \subset G} U(\phi(B)) \tag{1}$$

where $\phi(B)$ denotes the restriction of the configuration ϕ to the set B, and the potential $U(\phi(B))$ is a translationally invariant function. On the potential $U(\phi(B))$ we impose a natural condition, necessary for the existence of the thermodynamic limit:

$$\sum_{B \subset G: x \in B} |U(\phi(B))| < \text{const} \tag{2}$$

where the *const* does not depend on the configuration ϕ.

Let V be a finite volume in G. The concatenation of the configurations $\phi(V)$ and ϕ^i we denote by $\phi \diamond \phi^i : \phi \diamond \phi^i(x) = \phi(x)$ if $x \in V$, and $\phi \diamond \phi^i(x) = \phi^i(x)$ if $x \in G - V$. The finite-volume of the Gibbs distribution corresponding to the boundary conditions ϕ^i is:

$$\mathbf{P}^{\mathbf{i}}_{\mathbf{V}}(\phi|\phi^i) = \frac{\exp(-\beta(H(\phi \diamond \phi^i) - H(\phi^i)))}{\Xi(V, \phi^i)}$$

where β is the inverse temperature and the partition function $\Xi(V, \phi^i) = \sum_{\phi \in V} \exp(-\beta H_V(\phi \diamond \phi^i) - H(\phi^i))$. A probability measure $\mathbf{P}$ on the configuration space Φ^G is said to be an infinite-volume Gibbs measure if for each V and for $\mathbf{P}$ almost all ϕ^i in Φ^G we have:

$$\mathbf{P}(\phi(V)|\phi^i(G - V) = \mathbf{P}^{\mathbf{i}}_{\mathbf{V}}(\varphi|\phi^i) \tag{3}$$

where $\mathbf{P}(\phi(V)|\phi^i(G - V)$ is a conditional probability of $\phi(V)$ given $\phi^i(G - V)$.

For each $A \subset G$ the boundary $\delta(A)$ of A is the set of all the points not belonging to A but incident to A. Let ϕ be a fixed configuration:

$$\begin{aligned}
\exp(-\beta H(\phi)) &= \exp(-\beta H_0(\phi)) \exp(-\beta \sum_{B \subset G} U(\phi(B))) \\
&= \exp(-\beta H_0(\phi)) \prod_{B \subset G} \exp(-\beta U(\phi(B))) \\
&= \exp(-\beta H_0(\phi)) \prod_{B \subset G} (1 + \exp(-\beta U(\phi(B)) - 1) \\
&= \exp(-\beta H_0(\phi)) \prod_{B \subset G} (1 + \gamma_B)
\end{aligned} \tag{4}$$

where $\gamma_B = \exp(-\beta U(\phi(B))) - 1$. Now, by expanding (4) we get:

$$\exp(-\beta H(\phi)) = \exp(-\beta H_0(\phi)) \sum_{q \subset M} \prod \gamma_B \tag{5}$$

where M is the set of all $B \subset G$ except pairs of the nearest neighboring points. In other words, instead of one configuration ϕ with an interaction of $H(\phi)$ and a statistical weight of $\exp(-\beta H(\phi))$ we get infinitely many copies ϕ_q of the configuration (each copy ϕ_q is equipped with different sub-interactions q, and each q is the union of some subsets B) each with a statistical weight of $\exp(-\beta H_0(\phi)) \prod_{B \in q} \gamma_B$. The main point in this new

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representation is that if in any sub interaction-equipped configuration two points are not connected by an interaction element, then the two points are not interacting: We have 'got rid of' long-range interactions [18] and the Markov property holds: If we define Gibbs measures by using the weights (5) as in (3), then $\mathbf{P}(\phi_q(V)|\phi^i(G-V)$ only depends on $\phi^i(\delta V \cup q)$.

Suppose that ϕ'_p and ϕ''_q are two configurations equipped with the sub-interactions p and q, respectively. Let $D \subset G$ be the set of all points $x \in G$ on which configurations ϕ'_{q_1} and ϕ''_{q_2} disagree: $\phi'_p \neq \phi''_q$ Two points $x, y \in D$ are adjacent if $\operatorname{dist}(x, y) = 1$. Two points $x, y \in D$ are interaction connected if $x, y \in B \in p$, or $x, y \in B \in q$. A path from $\bar{x} \in D$ to $\bar{y} \in D$ is a sequence of vertices $x_1 = \bar{x}, x_2, \ldots, x_l = \bar{y}$, where for each $i = 1, 2, \ldots, l-1$ the points x_i and x_{i+1} are adjacent or interaction connected. We say that the set of points $D^* \subset D$ is a path of disagreement, if any two points $x, y \in D^*$ there is a path from x to y.

Let μ_1 and μ_2 be two Gibbs measures for the same interaction.

Theorem 1. *Suppose that the* $(\mu_1 \times \mu_2)(\phi'_p, \phi''_q)$ *probability of the event is an infinite path of disagreement corresponding to zero. Then* $\mu_1 = \mu_2$.

Proof. Let A be an arbitrary finite set of vertices and $\phi(A)$ be any configuration on A. In order to prove the theorem we have to prove that $\mu_1(\phi(A)) = \mu_2(\phi(A))$. For each pair (ϕ'_p, ϕ''_q) the cluster of disagreement $C_A = C(A, \phi', \phi'')$ containing A is the set A and all lattice points $x \in \mathbf{Z}^\nu$ for which there exists a path of disagreement connecting x and $A \cup \delta(A)$. Let $T : \Omega \times \Omega \to \Omega \times \Omega$ be the transformation which exchanges ϕ' and ϕ'' on C_A: $T(\phi'_p, \phi''_q) = (\psi'_p, \psi''_q)$ where

$$\psi'_p(x) = \begin{cases} \phi''_p(x) & \text{if } x \in C_A \\ \phi'_p(x) & \text{otherwise} \end{cases}$$

$$\psi''_q(x) = \begin{cases} \phi'_q(x) & \text{if } x \in C_A \\ \phi''_q(x) & \text{otherwise} \end{cases}$$

The transformation T is readily one-to-one. If there is no infinite disagreement cluster for pair (ϕ', ϕ'') then by Markov the property of Gibbs measures

$$(\mu_1 \times \mu_2)(\phi'_p, \phi''_q) = (\mu_1 \times \mu_2)(\psi'_p, \psi''_q). \tag{6}$$

Therefore, since C_A is finite with probability one, (5) is held with probability one. Finally, by (5):

$$\begin{aligned} \mu_1(\phi(A)) &= \sum_{(\phi'_p, \phi''_q):\phi'_p(A)=\phi(A)} (\mu_1 \times \mu_2)(\phi'_p \times \phi''_q) \\ &= \sum_{(\psi'_p, \psi''_q):\psi''_q(A)=\phi(A)} (\mu_1 \times \mu_2)(\psi'_p \times \psi''_q)) = \mu_2(\phi(A)) \end{aligned}$$

where the summation in $\sum_{(\psi'_p, \psi''_q):\psi''_q(A)=\phi(A)}$ is taken over by all the pairs of equipped configurations (ϕ'_p, ϕ''_q) such that there is no infinite path of disagreement connected to A and $\phi'_p(A) = \phi(A)$; the summation in $\sum_{(\psi'_p, \psi''_q):\psi''_q(A)=\phi(A)}$ takes over all the pairs of the equipped configurations (ψ'_p, ψ''_q) such that there is no infinite path of disagreement connected to A and $\psi''_q(A) = \phi(A)$. The proof is complete. □

3. Applications and final remarks

Now we apply the disagreement-percolation criterion to the ferromagnetic Ising model with long-range interaction on Z^2 with the following Hamiltonian:

$$H(\phi) = -\sum_{x,y\in Z^2,\mathrm{dist}(x,y)=1} \phi(x)\phi(y) - \sum_{x,y\in Z^2,\mathrm{dist}(x,y)>1} c^{-\mathrm{dist}(x,y)}\phi(x)\phi(y) = H_0 + H_1 \tag{7}$$

Theorem 2. *There are positive constants β_0 and c_0 as for all non-negative $\beta < \beta_0$ and $c < c_0$ the model (7) has a unique limiting Gibbs state.*

Proof. The uniqueness of Gibbs states will be proved by employing theorem 1. Let μ_1 and μ_2 be two Gibbs measures corresponding to the Hamiltonian (7). In order to do this we will prove that the $(\mu_1 \times \mu_2)$ probability of the event is an infinite path of zero disagreement. Let $\mathbf{P_V^i}(\varphi|\phi^i)$ be the finite volume Gibbs distribution corresponding to the Hamiltonian H_0, given a finite volume $V \subset Z^2$ and boundary conditions ϕ^i. Let $x' \in V$. Since the model (7) is ferromagnetic there is a β_1, such that for all $\beta < \beta_1$:

$$\mathbf{P_V^i}(\phi(x') = 1|\phi^i) \leqslant \frac{\mathrm{e}^{4\beta}}{\mathrm{e}^{4\beta} + \mathrm{e}^{-4\beta}} < 0.501 \tag{8}$$

The first inequality in (8) is due to Griffiths inequality (the probability of $+1$ is maximal when all other spins are $+1$). The second inequality is due to the fact that $\frac{\mathrm{e}^{4\beta}}{\mathrm{e}^{4\beta}+\mathrm{e}^{-4\beta}} = \frac{1}{2}$ at $\beta = 0$ and the dependence on β is continuous. A similar inequality holds for the probability of $\mathbf{P_V^i}(\phi(x') = -1|\phi^i)$. Since the interaction in the model (7) is pairwise the interaction elements are $\gamma_B = \gamma(x,y) = \exp(\pm\beta c^{-\mathrm{dist}(x,y)}) - 1$. There is β_2, so that for $\beta < \beta_2$ we have $\gamma(x,y) \leqslant c^{-\mathrm{dist}(x,y)}$. We choose $\beta_0 = \min(\beta_1, \beta_2)$. If two points of disagreement are connected by some interaction element we will assume that these two points are connected through the shortest path of the neighboring lattice points. All the lattice points lying on the path will be called interaction points. If the path connects points $x' = (a', b')$ and $y' = (a'', b'')$ and $|a' - a''| = t_1$ and $|b' - b''| = t_2$, then by arithmetic-quadratic mean inequality $t_1 + t_2 \leqslant 2\sqrt{t_1^2 + t_2^2} = \sqrt{2}|\rho|$. By a uniform 'distribution' of the weight $c^{-\mathrm{dist}(x,y)}$ to all $t = t_1 + t_2$ interaction points we get $\gamma(x,y) \leqslant (\sqrt[\sqrt{2}]{c})^t$. This means that the statistical weight of the interaction between the two points over the interaction path is the product of the weights of the interaction points, each not exceeding $\sqrt[\sqrt{2}]{c}$. Thus, by Peierls' argument the probability that a given lattice point is an interaction point is also at most $\sqrt[\sqrt{2}]{c}$ (let x be a lattice point, to each configuration equipped with sub-interactions for which x is an interaction point in the numerator we correspond a configuration equipped with sub-interactions for which x is not an interaction point in the denominator). Since we are considering a pair of configurations, the probability that a lattice point is an interaction point is at most $1-(1-c^{\frac{1}{\sqrt{2}}})^2$. Choosing a c_0 such that for all $c < c_0$ we have $1 - (1 - c^{\frac{1}{\sqrt{2}}})^2 < 0.05$. Now we are ready to estimate the disagreement paths. By (8) the probability that the lattice point x is a point of disagreement is at most $2 \cdot 0.501 \cdot 0.501 < 0.5021$. Since the path of disagreement consists of disagreement lattice points and possible interaction points then the probability of an infinite path of disagreement is no greater than the probability of site percolation with $p = 0.5021 + 0.05 = 0.5521$.

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Since the site percolation threshold for Z^2 exceeds 0.556 (see [23]) the probability of infinite disagreement-percolation is zero and by theorem 1 we are done. $\square$

Theorem 1 is proved by the method of coupling using Gibbs measures. Different methods involving coupling ideas were previously [20–22]. The term 'disagreement' we adopt from [21] and [22]. The similar disagreement uniqueness condition for models with an interaction of neighboring spins was firstly formulated in [21]. The crucial transformation of $T(\phi', \phi'') = (\psi', \psi'')$ in the proof of theorem 1 is also taken from [21]. In [19] the uniqueness of Gibbs states in 1D with long-range antiferromagnetic Ising models has been proved again by the method of coupling two independent realizations. Afterwards this method was generalized for models with a unique ground state [18]. The method employed in [18, 19] also uses the finite-volume version of the transformation $T(\phi', \phi'') = (\psi', \psi'')$. In [18,19] the undesirable products of coupling are clusters connecting the set A with the boundary conditions. When these clusters are negligible, the two Gibbs states become completely continuous with respect to each other and coincide. Since the models in [18, 19] have a unique ground state, the disagreement clusters are defined as clusters not coinciding with a unique ground state. Thus, theorem 1 generalizes theorem 1 of [18] for models with long-range interaction (in a case when only neighboring spin variables interact, theorem 1 becomes theorem 1 of [21]) and generalizes theorem 2 of [18] for models having more then one ground state. That is why in this section we have applied the disagreement-percolation uniqueness criterion (theorem 1) to models (7) having more than one ground state.

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