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Maskin-monotonic scoring rules

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Abstract We characterize which scoring rules are *Maskin-monotonic* for each social choice problem as a function of the number of agents and the number of alternatives. We show that a scoring rule is *Maskin-monotonic* if and only if it satisfies a certain unanimity condition. Since scoring rules are *neutral*, *Maskin-monotonicity* turns out to be equivalent to *Nash-implementability* within the class of scoring rules. We propose a class of mechanisms such that each *Nash-implementable* scoring rule can be implemented via a mechanism in that class. Moreover, we investigate the class of *generalized scoring rules* and show that with a restriction on score vectors, our results for the standard case are still valid.

1 Introduction

Consider a group of alternatives and a group of voters. Voters have preference rankings over alternatives, and they have to make a choice. One common way of doing so is to choose the alternative(s) that is (are) top-ranked by most voters. This method is known as the "plurality rule". There is another common way which takes into consideration not only the top-ranked alternatives, but the whole rankings of the voters. For each voter, assign to each alternative a "score" equal to its rank from the bottom in that voter's preference ranking. Then, choose the alternative(s) that achieves(achieve) the highest total score. This method is known as "Borda's rule" (Borda 1781). One common feature of plurality rule and Borda's rule is that they are based on a prespecified number

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sequence. If for each voter we assign those numbers to the alternatives according to his preference ranking, the alternative(s) achieving the highest total score is (are) chosen. By changing the number sequence, we obtain other rules. These rules are called "scoring rules".

Here, we are interested in identifying those scoring rules that satisfy a certain property called *Maskin-monotonicity*. In implementation theory, *Maskin-monotonicity* is a central concept mainly because it is a necessary condition for *Nash-implementability* (Maskin 1977). It requires the following. Consider a preference profile and an alternative chosen at this profile. Consider another preference profile such that the position of the chosen alternative relative to each of the other alternatives either improves or stays the same. Then that alternative should still be chosen at the second profile. *Maskin-monotonicity* is a necessary condition for Nash-implementability, but unfortunately many widely used rules are not *Maskin-monotonic*. In fact, when there are at least three alternatives, an *onto* and *single-valued* rule defined on the "full domain" of preference profiles is *Maskin-monotonic* if and only if it is *dictatorial* (Muller and Satterthwaite 1977). Also for each of the two best-known scoring rules, namely plurality rule and Borda's rule, one can specify a number of alternatives and a number of voters such that these rules are not *Maskin-monotonic*. In fact, Erdem and Sanver (2005) shows that when there are three voters and three alternatives, no scoring rule is *Maskin-monotonic*. Here, we characterize the *Maskin-monotonic* scoring rules for each problem as a function of the number of alternatives and the number of voters. Moreover, we give the number of *Maskin-monotonic* rules as a function of the number of alternatives and the number of voters.

We first show that when the number of alternatives does not exceed the number of voters, no scoring rule is *Maskin-monotonic*. Given a score vector s, we define $k^*(s)$ as the smallest k satisfying $s_k > s_{k+1}$. We show that the scoring rule associated with s is *Maskin-monotonic* if and only if $k^*(s) > \frac{m(n-1)}{n}$, where m is the number of alternatives and n is the number of voters (Theorem 1). Moreover, within the class of scoring rules, *Maskin-monotonicity* is equivalent to a certain condition, which requires an alternative to be chosen if and only if that alternative achieves the maximal possible score.

We also consider the *Nash-implementability* of scoring rules and propose a class of mechanisms such that each *Nash-implementable* scoring rule can be implemented via a mechanism in that class.

Finally, we study "generalized" scoring rules, where there are possibly different score vectors associated with voters. Here, by imposing the restriction that for each voter the scores assigned to his first-best choice and his second-best choice be equal, we obtain results similar to the ones we obtained for standard scoring rules.

2 Preliminaries

Let $A = \{a_1, \ldots, a_m\}$ be a set of alternatives and $N = \{1, \ldots, n\}$ a set of voters such that $m, n \geq 3$, and $(m, n) \neq (3, 4)$. Let $\mathcal{L}(A)$ be the set of linear orders[1] on A. A

[1] A linear order is a transitive, anti-symmetric and complete binary relation.

preference profile is an n-tuple of linear orders on A, $R = (R_1, \ldots, R_n) \in \mathcal{L}(A)^N$. Let 2^A be the set of all subsets of A. A social choice rule, or simply a *rule*, is a function $F : \mathcal{L}(A)^N \to 2^A \backslash \emptyset$.

A *score vector* is an m-tuple $s = (s_1, \ldots, s_m) \in \mathbb{R}^m$ such that $s_1 > s_m$, and for each $i \in \{1, \ldots, m-1\}$, $s_i \geq s_{i+1}$. For each $m \in \mathbb{N}$, let $\mathcal{S}^m$ denote the set of score vectors.

For each $a \in A$ and each $R \in \mathcal{L}(A)^N$, let $\boldsymbol{\sigma(a, R_i)}$ denote the rank of a in voter i's ordering, i.e. $\sigma(a, R_i) = |\{b \in A \mid b\ R_i\ a\}|$.

Let $s \in \mathcal{S}^m$. The *scoring rule associated with* $\boldsymbol{s}$, $\boldsymbol{F^s}$, associates with each $R \in \mathcal{L}(A)^N$ the set

$$F^s(R) \equiv \left\{a \in A \mid \text{ for each } \quad b \in A, \quad \sum_{i \in N} s_{\sigma(a, R_i)} \geq \sum_{i \in N} s_{\sigma(b, R_i)}\right\}.$$

Note that two different score vectors can be associated with the same scoring rule.

For each $a \in A$, each $R \in \mathcal{L}(A)^N$, and each $i \in N$, let $L(R_i, a) = \{b \in A \mid a\ R_i\ b\}$ denote the *lower contour set of* $\boldsymbol{R_i}$ *at* $\boldsymbol{a}$. Also let $MT(R, a) = \{R' \in \mathcal{L}(A)^N \mid \forall i \in N : L(R_i, a) \subseteq L(R'_i, a)\}$. A rule $F \in \mathcal{F}$ is ***Maskin-monotonic*** if and only if for each $R \in \mathcal{L}(A)^N$, each $a \in F(R)$, and each $R' \in MT(R, a)$, we have $a \in F(R')$. Let $\mathcal{M}$ be the set of all *Maskin-monotonic* rules.

3 Results

Proposition 1 *If $m \leq n$, no scoring rule is Maskin-monotonic.*

Proof Let $s \in \mathcal{S}^m$.

Case 1 $\boldsymbol{m = n}$. Let $R \in \mathcal{L}(A)^N$ be such that for each $k \in \{1, \ldots, m-1\}$, $a_k\ R_1\ a_{k+1}$, and for each $i \in N \backslash \{1\}$, R_i is obtained from R_{i-1} by moving the top-ranked alternative to the bottom.

R_1	R_2	$\ldots$	R_{m-1}	R_m
a_1	a_2	$\ldots$	a_{m-1}	a_m
a_2	a_3	$\ldots$	a_m	a_1
a_3	a_4	$\ldots$	a_1	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_m	a_1	$\ldots$	a_{m-2}	a_{m-1}

Note that $F^s(R) = A$. Since $s_1 > s_m$, and for each $i \in \{1, \ldots, m-1\}$, $s_i \geq s_{i+1}$, there is $k \in \{1, \ldots, m-1\}$ such that $s_k > s_{k+1}$. Let $R' \in \mathcal{L}(A)^N$ be obtained from R by only interchanging a_k and a_{k+1} in R_1. Since $m \geq 3$, there is $t \in \{1, \ldots, m\} \backslash \{k, k+1\}$. Note that $R' \in MT(R, a_t)$. Yet $a_t \notin F^s(R')$. Thus, $F^s \notin \mathcal{M}$.

Case 2 $\boldsymbol{m < n}$. Let $p \in \mathbb{Z}^+$, $q \in \{0, \ldots, m-1\}$, and $n = pm + q$. Let $R' \in \mathcal{L}(A)^N$ be such that for each $i \in \{1, \ldots, m\}$ and each $j \in \{0, \ldots, p-1\}$, $R'_{i+j.m} = R_i$. If $q > 0$, then for each $i \in \{1, \ldots, q\}$, let $R'_{pm+i} \in \mathcal{L}(A)^N$ be such that

(a) If i is odd,

$$\sigma(a_1, R'_{pm+i}) = 1,$$
$$\sigma(a_3, R'_{pm+i}) = 2,$$
$$\sigma(a_2, R'_{pm+i}) = 3,$$

and for each $j \in \{4, \ldots, m\}$, $\sigma(a_j, R'_{pm+i}) = j$,

(b) If i is even,

$$\sigma(a_3, R'_{pm+i}) = 1,$$
$$\sigma(a_1, R'_{pm+i}) = 2,$$
$$\sigma(a_2, R'_{pm+i}) = 3,$$

and for each $j \in \{4, \ldots, m\}$, $\sigma(a_j, R'_{pm+i}) = j$.

Note that the orderings between R'_m and R'_{pm+1} are obtained by replicating the first m orderings.

R'_1	R'_2	$\ldots$	R'_m	$\ldots$	R'_{pm+1}	R'_{pm+2}	R'_{pm+3}	$\ldots$
a_1	a_2	$\ldots$	a_m	$\ldots$	a_1	a_3	a_1	$\ldots$
a_2	a_3	$\ldots$	a_1	$\ldots$	a_3	a_1	a_3	$\ldots$
a_3	a_4	$\ldots$	a_2	$\ldots$	a_2	a_2	a_2	$\ldots$
a_4	a_5	$\ldots$	a_3	$\ldots$	a_4	a_4	a_4	$\ldots$
a_5	a_6	$\ldots$	a_4	$\ldots$	a_5	a_5	a_5	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$
a_m	a_1	$\ldots$	a_{m-1}	$\ldots$	a_m	a_m	a_m	$\ldots$

When $q = 0$, i.e. when m divides n, by the same argument as in Case 1, $F \notin \mathcal{M}$. Thus, suppose $q > 0$.

Claim 1 For each $k \in \{1, \ldots, m-1\}$, there is $i \in N$ such that $\sigma(a_2, R'_i) = k$ and $\sigma(a_3, R'_i) = k + 1$. We omit the obvious proof.

Claim 2 If $s_1 > s_2$ and F^s is *Maskin-monotonic*, then there is $k \in \{2, \ldots, m-1\}$ such that $s_k > s_{k+1}$. To see this, note that if $s_1 > s_2 = s_3 = \cdots = s_m$, then F^s is plurality rule. However, when $m \geq 3$, $n \geq 3$, and $(m, n) \neq (3, 4)$, plurality rule is not *Maskin-monotonic*. To see that, suppose that $4 \neq n \geq 3$. If n is odd, let $R \in \mathcal{L}(A)^N$ be such that each $i \in \{1, 2, \ldots, \frac{n-1}{2}\}$ top ranks a_1, each $j \in \{\frac{n-1}{2} + 1, \ldots, n-1\}$ top ranks a_2, and voter n top ranks a_3 and second ranks a_2. Let $R' \in \mathcal{L}(A)^N$ be obtained from R by only interchanging the positions of a_2 and a_3 in voter n's ordering. Note that $a_1 \in F^s(R)$ and $R' \in MT(R, a_1)$. Yet $a_1 \notin F^s(R')$. Thus, $F^s \notin \mathcal{M}$. If n is even, let $R \in \mathcal{L}(A)^N$ be such that each $i \in \{1, 2, \ldots, \frac{n-2}{2}\}$ top ranks a_1, each $j \in \{\frac{n-2}{2} + 1, \ldots, n-2\}$ top ranks a_2, and voters $n-1$ and n top rank a_3 and second rank a_2. Let $R' \in \mathcal{L}(A)^N$ be obtained from R by only interchanging the positions of a_2 and a_3 in voters $n-1$ and n's orderings. Note that $a_1 \in F^s(R)$ and $R' \in MT(R, a_1)$. Yet $a_1 \notin F^s(R')$. Thus, $F^s \notin \mathcal{M}$. One can easily show that when $(m, n) = (3, 4)$, plurality rule is *Maskin-monotonic*. This is the reason we exclude this pair.

Now, suppose that $s_1 > s_2$. Suppose that voter n top ranks a_3, i.e. q is even. Then, clearly $F^s(R') = \{a_1, a_3\}$. Let $R'' \in \mathcal{L}(A)^N$ be obtained from R' by only interchanging a_2 and a_3 in voter 2's ordering. Now, although $R'' \in MT(R', a_1)$, clearly $a_1 \notin F^s(R'') = \{a_3\}$. So, suppose that voter n top ranks a_1, i.e. q is odd. Now, $a_1 = F^s(R')$ and $\sum_{i\in N} s_{\sigma(a_1,R_i)} = \sum_{i\in N} s_{\sigma(a_3,R_i)} + (s_1 - s_2)$. From Claim 2, there is $k \in \{2, \dots, m-1\}$ such that $s_k > s_{k+1}$. Then, from Claim 1, there is $i \in N$ such that $\sigma(a_2, R'_i) = k$ and $\sigma(a_3, R'_i) = k+1$. Let $R''' \in \mathcal{L}(A)^N$ be obtained from R' by only interchanging a_2 and a_3 in agent 2's and agent i's orderings. Now, $\sum_{i\in N} s_{\sigma(a_1,R_i)} = \sum_{i\in N} s_{\sigma(a_3,R_i)} - (s_k - s_{k+1})$. Note that $R''' \in MT(R', a_1)$. Yet $a_1 \notin F^s(R''')$. Thus, $F^s \notin \mathcal{M}$.

Suppose that $s_1 = s_2$. Then, $a_1 \in F^s(R')$ and $a_3 \in F^s(R')$. Now, since $s_1 > s_m$, there is $k \in \{2, \dots, m-1\}$ such that $s_k > s_{k+1}$. Also, from Claim 1, there is $i \in N$ such that $\sigma(a_2, R'_i) = k$ and $\sigma(a_3, R'_i) = k+1$. Let $R'' \in \mathcal{L}(A)^N$ be obtained from R' by only interchanging a_2 and a_3 in voter i's ordering. Note that $R'' \in MT(R', a_1)$. Yet $a_1 \notin F^s(R'') = a_3$. Thus, $F^s \notin \mathcal{M}$. □

Lemma 1 *Let $m > n$. Let $s \in \mathcal{S}^m$. If F^s is Maskin-monotonic, then for each $k \leq n$, $s_1 = s_k$.*

Proof Suppose not. Let R be the profile defined in Case 1 of Proposition 1. Let $R' \in \mathcal{L}(A)^N$ be such that for each $i \in N$, the highest-ranked n alternatives according to R'_i are the same as according to R, and for each $k \in \{n+1, \dots, m\}$, and each $i \in N$, $\sigma(a_k, R'_i) = k$.

R'_1	R'_2	$\dots$	R'_{n-1}	R'_n
a_1	a_2	$\dots$	a_{n-1}	a_n
a_2	a_3	$\dots$	a_n	a_1
a_3	a_4	$\dots$	a_1	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_n	a_1	$\dots$	a_{n-2}	a_{n-1}
a_{n+1}	a_{n+1}	$\dots$	a_{n+1}	a_{n+1}
a_{n+2}	a_{n+2}	$\dots$	a_{n+2}	a_{n+2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_{m-1}	a_{m-1}	$\dots$	a_{m-1}	a_{m-1}
a_m	a_m	$\dots$	a_m	a_m

Here, for each $k \in \{1, \dots, n\}$, $a_k \in F(R')$. But note that there is $k \in \{1, \dots, n-1\}$ such that $s_k > s_{k+1}$. Now, let $R'' \in \mathcal{L}(A)^N$ be obtained from R' by only interchanging a_k and a_{k+1} in R_1. Since $n \geq 3$, there is $t \in \{1, \dots, n\} \backslash \{k, k+1\}$. But now, $R'' \in MT(R', a_t)$. Yet $a_t \notin F(R'')$. Thus, $F^s \notin \mathcal{M}$. □

For each $s \in \mathcal{S}^m$, let $k^*(s) \in \{1, \dots, m-1\}$ be the smallest integer s such that $s_k > s_{k+1}$, i.e. $k^*(s) = \min \left\{k \in \{1, \dots, m-1\} \mid s_k > s_{k+1}\right\}$.

Theorem 1 *Let $s \in \mathcal{S}^m$. The rule F^s is Maskin-monotonic if and only if $k^*(s) > \frac{m(n-1)}{n}$.*

Proof ($\Longleftarrow$) (For notational simplicity, we write k^* instead of $k^*(s)$). Suppose $k^* > \frac{m(n-1)}{n}$, i.e. $k^*n > mn - m$, i.e. $m > n(m-k^*)$. Note that for each $R \in \mathcal{L}(A)^N$, $n(m-k^*)$ is the maximal number of alternatives whose rank is greater than k^* for at least one voter. Since $m > n(m-k^*)$, at each $R \in \mathcal{L}(A)^N$ there is at least one alternative whose rank is less than k^* for all voters. Moreover, since $s_1 = s_2 = \cdots = s_{k^*}$, that alternative also achieves the highest possible score, namely ns_1. But then, for each $R \in \mathcal{L}(A)^N$ and each $a \in F^s(R)$, a achieves ns_1, and for each $R' \in MT(R, a)$, we have $a \in F^s(R')$. Thus, $F^s \in \mathcal{M}$.

($\Longrightarrow$) Suppose that $k^* \leq \frac{m(n-1)}{n}$, i.e $m \leq n(m-k^*)$. By Proposition 1, we have $m > n$, and by Lemma 1, we have $k^* \geq n \geq 3$. Let $R \in \mathcal{L}(A)^N$ be such that $\sigma(a_1, R_1) = k^* + 1, \sigma(a_2, R_1) = k^* + 2, \ldots, \sigma(a_{m-k^*}, R_1) = m$, and there is $q \in \{1, \ldots, m\}$ such that $\sigma(a_{m-k^*+1}, R_2) = k^* + 1, \ldots, \sigma(a_q, R_n) = m$.

$$
\begin{array}{c|cccc}
 & R_1 & R_2 & \ldots & R_n \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots \\
k^*+1 \rightarrow & a_1 & a_{m-k^*+1} & \ldots & \ldots \\
k^*+2 \rightarrow & a_2 & a_{m-k^*+2} & \ldots & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
m \rightarrow & a_{m-k^*} & \ldots & \ldots & a_q
\end{array}
$$

Now, let $R' \in \mathcal{L}(A)^N$ be obtained from R by only moving a_1 to rank $k^* + 1$ in the ordering of each $i \in N$ such that $\sigma(a_1, R_i) > k^*$.

Suppose that $a_1 \in F^s(R')$. Let $R'' \in \mathcal{L}(A)^N$ be obtained from R' by only moving a_1 to the top and a_m to second position in the orderings of all voters except for voter 1. Note that $\sigma(a_1, R_1) = k^* + 1$ and $\sigma(a_m, R_1) < k^* + 1$. Then, $R'' \in MT(R', a_1)$. Yet $a_1 \notin F^s(R'')$. Thus, $F^s \notin \mathcal{M}$.

Suppose that there is $k \in \{2, \ldots, m\}$ such that $a_k \in F^s(R')$. Note that there is $i \in N$ such that $\sigma(a_k, R'_i) > k^*$ and $\sigma(a_1, R'_i) < \sigma(a_k, R'_i)$. Let $R'' \in \mathcal{L}(A)^N$ be obtained from R' by only moving a_k to the top and a_1 to second position in the orderings of all voters except for voter i and moving a_1 to the top in the ordering of voter i. Note that $R'' \in MT(R', a_k)$. Yet $a_k \notin F^s(R'')$. Thus, $F^s \notin \mathcal{M}$.

We have established $F^s(R) = \emptyset$, which is not possible since the sets of alternatives and voters are finite, and there has to be an alternative that achieves the highest score. □

Theorem 1 suggests that, if a scoring rule is *Maskin-monotonic*, then it chooses the alternatives achieving the maximal possible score and only these. Let $s \in \mathcal{S}^m$. The scoring rule $F^s \in \mathcal{S}$ is ***unanimous*** if and only if for each $R \in \mathcal{L}(A)^N$,

$$a \in F^s(R) \Leftrightarrow \sum_{i \in N} s_{\sigma(a, R_i)} = n.s_1$$

That is, F^s chooses an alternative if and only if it achieves the maximal possible score.

Corollary 1 *Let $s \in \mathcal{S}^m$. The rule F^s is Maskin-monotonic if and only if F^s is unanimous.*

Proof Suppose that F^s is *unanimous*. Let $R \in \mathcal{L}(A)^N$, $a \in F^s(R)$. Now, since F^s is *unanimous*, a achieves the maximal possible score. Then, at each $R' \in MT(R, a)$, $a \in F(R')$. Thus, $F \in \mathcal{M}$.

Suppose that $F^s \in \mathcal{M}$. By Theorem 1, $k^*(s) > \frac{m(n-1)}{n}$, i.e. $k^*(s)n > mn - m$, i.e. $m > n(m - k^*(s))$. Note that at each $R \in \mathcal{L}(A)^N$, $n(m - k^*(s))$ is the maximal number of alternatives whose rank is greater than $k^*(s)$. Since $m > n(m - k^*(s))$, then at each $R \in \mathcal{L}(A)^N$ there is at least one alternative whose rank is less than $k^*(s)$ at each voters' ordering. Moreover, since $s_1 = s_2 = \cdots = s_{k^*(s)}$, that alternative achieves the maximal possible score ns_1. Thus, F^s is *unanimous*. □

As one would expect, not too many scoring rules are *Maskin-monotonic*. In fact when $m \leq n$, no scoring rule is *Maskin-monotonic*. When $n < m \leq 2n$, only one scoring rule is *Maskin-monotonic*. It is the "antiplurality rule", associated with the score vector $(1, 1, \ldots, 1, 0)$. When $2n < m \leq 3n$, only two scoring rules are *Maskin-monotonic*. One of them is the rule associated with $(1, 1, \ldots, 1, 1, 0)$, and the other is the rule associated with $(1, 1, \ldots, 1, 0, 0)$. In fact, if we define a k-plurality rule for $k \in \{1, \ldots, m-1\}$ as the scoring rule associated with the score vector $(1, 1, \ldots, 1, 0, 0, \ldots, 0)$ where the first k entries are 1, the set of k-plurality rules such that $k > \frac{m(n-1)}{n}$ is the same as the set of *unanimous* and thus *Maskin-monotonic* scoring rules. Note that when $k > \frac{m(n-1)}{n}$, the k-plurality rule is associated with each $s \in \mathcal{S}^m$ such that $s_1 = s_2 = \cdots = s_k > s_{k+1}$. Thus, we obtain the number of *Maskin-monotonic* scoring rules:

Corollary 2 *The number of Maskin-monotonic scoring rules is* $\max\{0, \lfloor \frac{m-0.1}{n} \rfloor\}$.[2]

4 Nash-implementation of scoring rules

A *mechanism* is an ordered pair $G = (M, \pi)$ where $M = \Pi_{i \in N} M_i$ is a nonempty *strategy space* and $\pi : M \to A$ an *outcome function*. A mechanism *Nash-implements* $F \in \mathcal{F}$ if and only if for each $R \in \mathcal{L}(A)^N$, the set of Nash-equilibrium outcomes of the game (G, R) coincides with $F(R)$, i.e. $\pi(NE(G, R)) = F(R)$. Let $s \in \mathcal{S}^m$. Since F^s is *neutral* and there are at least three voters, F^s is *Nash-implementable* if and only if it is *Maskin-monotonic* (Maskin 1977).

Consider the class of mechanisms $\{G(t)\}_{t=1,\ldots,m-1}$ with the following strategy space and outcome function. Each voter i announces a linear ordering of $m - t + 1$ alternatives, say R^i, and a positive integer, say z_i. If the same alternative is top-ranked in the orderings of all the voters, that alternative is chosen. If $(n-1)$ of the voters top rank the same alternative but some voter $j \in N$ top ranks a different alternative, the $[z_j \pmod{m - t + 1}]$'th best alternative in the ordering of voter $j + 1$ is chosen. If $j = n$, let $j + 1 = 1$. If there are at least three different top-ranked alternatives, the alternative that is top-ranked by the voter who announced the highest integer is choosen. (Ties are broken on behalf of the voter with the smallest index.)

Proposition 2 *Let* $s \in \mathcal{S}^m$. *If* F^s *is Maskin-monotonic, then mechanism* $G(k^*)$ *Nash-implements* F^s.

[2] $\lfloor x \rfloor$ denotes the maximal integer that does not exceed x.

Proof Let $R \in \mathcal{L}(A)^N$. Let $a \in F^s(R)$. By Theorem 1, for each $i \in N$, $\sigma(a, R_i) \leq k^*$. Now, consider the following strategy profile. Each voter $i \in N$ announces a linear ordering of $m - k^* + 1$ alternatives, say R^i, such that: a is top-ranked at R^i; and for each $b \in A \backslash \{a\}$, b is included in R^i if and only if $\sigma(b, R_{i-1}) \geq k^* + 1$ (in case $i = 1$, let $i - 1 = n$). Each $i \in N$ also announces $z_i \in \mathbb{Z}^+$. First of all, since a is top-ranked by each voter, a is chosen at this strategy profile. Moreover, when the strategies of the other voters are fixed, by changing his strategy, each voter i can only ensure an alternative which has a rank greater than k^* at R_i to be chosen. However, alternative a's rank at R_i is less than or equal to k^*. So, this strategy profile is a Nash-equilibrium at which a is chosen.

Now, let $m \in M$ be such that m is a Nash-equilibrium of $G(k^*)$ at R, and $a = \pi(m)$. For each $i \in N$, let R^i be the linear ordering of $m - k^* + 1$ alternatives announced by voter i. First, suppose that for each $i \in N$, a is top-ranked in R^i. Suppose that $a \notin F^s(R)$. Then, from Theorem 1, there is $i \in N$ such that $\sigma(a, R_i) > k^*$. Since each voter announces an ordering of $m - k^* + 1$ alternatives, there is $b \in A$ from among the alternatives announced by voter $i+1$ such that $\sigma(b, R_i) < k^*$. By changing his top-ranked alternative in R^i and by changing z_i, voter i can ensure that b is chosen, which contradicts m being a Nash-equilibrium. Thus, $a \in F^s(R)$.

Now, suppose that a is top-ranked in the orderings announced by each voter except voter j. Suppose that $a \notin F^s(R)$. Then, from Theorem 1, there is $i \in N$ such that $\sigma(a, R_i) > k^*$. If $j = i$, the previous reasoning directly applies. Suppose that $j \neq i$. Now, voter i can top rank his best alternative at R, and changing z_i, and make his top alternative at R chosen, which contradicts m being a Nash-equilibrium. For the last case, suppose there is at least three different top-ranked alternatives, and $a \notin F^s(R)$. Again clearly, there is $i \in N$ such that $\sigma(a, R_i) > k^*$. By changing z_i, voter i can ensure that his top alternative is chosen at R. This contradicts m being a Nash-equilibrium. Thus, $\pi(NE(G[R])) = F^s(R)$, and $G(k^*)$ Nash-implements F^s. □

5 Generalized scoring rules

So far, we have considered scoring rules for which the score vector is the same for each voter. Now, we will define "generalized scoring rules" induced by score vectors that may vary from voter to voter. As opposed to scoring rules that are *anonymous*, that is, treats voters symmetrically, generalized scoring rules allow us to favor some voters. We obtain results similar to the ones for standard scoring rules, by imposing a restriction on score vectors. The restriction is that, for each voter, the scores associated with his top-ranked and second-ranked alternatives in his score vector are equal.

Let $s^1, \ldots, s^n \in \mathcal{S}^m$. A *generalized score vector* is a family of score vectors, $S = (s^1, \ldots, s^n) \in \mathbb{R}^{m \times n}$. For each $m, n \in \mathbb{N}$, let $\mathcal{S}^{(m,n)}$ be the set of generalized score vectors. The *generalized scoring rule associated with* $\boldsymbol{S \in \mathcal{S}^{(m,n)}}$, $\boldsymbol{F^S}$, is defined by setting for each $R \in \mathcal{L}(A)^N$,

$$F^S(R) = \left\{ a \in A \mid \text{ for each } \quad b \in A, \quad \sum_{i \in N} s^i_{\sigma(a,R_i)} \geq \sum_{i \in N} s^i_{\sigma(b,R_i)} \right\}.$$

For each $S \in \mathcal{S}^{(m,n)}$, each $i \in N$, let $k^*(s^i) = \min\left\{k \in \{1, \ldots, m-1\} \mid s^i_k > s^i_{k+1}\right\}$.

Theorem 2 *Let $S \in \mathcal{S}^{(m,n)}$ be such that for each $i \in \{1, \ldots, n\}$, $s^i_1 = s^i_2$. The rule F^S is Maskin-monotonic if and only if $\sum_{i\in N}(m - k^*(s^i)) < m$.*

Proof ($\Longleftarrow$) (For notational simplicity, for each $i \in N$, we write k^*_i instead of $k^*(s^i)$). Suppose $\sum_{i\in N}(m - k^*_i) < m$. Note that for each $R \in \mathcal{L}(A)^N$, $\sum_{i\in N}(m - k^*_i)$ is the maximal number of alternatives whose rank is greater than k^*_i for at least one voter i. Since $\sum_{i\in N}(m - k^*_i) < m$, at each $R \in \mathcal{L}(A)^N$ there is at least one alternative whose rank is less than k^*_i for each $i \in N$. Moreover, that alternative also achieves the highest possible score, namely $\sum_{i\in N} s^i_1$. But then, for each $R \in \mathcal{L}(A)^N$ and each $a \in F^S(R)$, a achieves $\sum_{i\in N} s^i_1$, and for each $R' \in MT(R, a)$, we have $a \in F^S(R')$. Thus, $F^S \in \mathcal{M}$.

($\Longrightarrow$) We prove the contrapositive statement. Suppose that $\sum_{i\in N}(m - k^*_i) > m$. Note that there is a profile, say $R \in \mathcal{L}(A)^N$, such that for each alternative $a \in A$, there is $i \in N$ such that $\sigma(a, R_i) > k^*_i$. Let $a \in F^S(R)$. Let $j \in N$ be such that $\sigma(a, R_j) > k^*_j$. Let b be the alternative that is top-ranked at R_j, i.e. $\sigma(b, R_j) = 1$. Let $R' \in \mathcal{L}(A)^N$ be obtained from R by moving a to top and b to second rank at each agents' ordering except for R_j. Note that $R' \in MT(R, a)$. Yet $a \notin F^S(R')$. Thus, $F^S \notin \mathcal{M}$. □

We can also modify the definition of a *unanimous* scoring rule. Let $S \in \mathcal{S}^{(m,n)}$. The rule F^S is *unanimous* if and only if for each $R \in \mathcal{L}(A)^N$,

$$a \in F^S(R) \Leftrightarrow \sum_{i\in N} s^i_{\sigma(a,R_i)} = \sum_{i\in N} s^i_1.$$

Corollary 3 *Let $S \in \mathcal{S}^{(m,n)}$ be such that for each $i \in \{1, \ldots, n\}$, $s^i_1 = s^i_2$. The rule F^S is Maskin-monotonic if and only if F^S is unanimous.*

Note that for a generalized scoring rule, we require for each $i \in N$, $s^i_1 = s^i_2$, to obtain our characterization. Although this is a necessary condition for *Maskin-monotonicity* of standard scoring rules, there are *Maskin-monotonic* generalized scoring rules that do not satisfy this condition. As an example, let $(s^1, \ldots, s^n) \in \mathcal{S}^{(m,n)}$ be such that $s^1_1 + \sum_{j\in\{2,\ldots,n\}} s^j_m > s^1_2 + \sum_{j\in\{2,\ldots,n\}} s^j_1$. Now, F^S is the dictatorial rule where voter 1 is the dictator, which is clearly *Maskin-monotonic* although $s^1_1 \neq s^1_2$. To see that this fact is not special to dictatorial rules, one can also consider the generalized scoring rule induced by $((2, 1, 0), (1, 0, 0), (2, 1, 0))$ which is *Maskin-monotonic* but is not a dictatorial rule while $s^i_1 \neq s^i_2$ for each $i \in \{1, 2, 3\}$.

6 Related literature

Two results in the literature are closely related to ours.

1. Given a score vector s, if $(m \geq 3, n = 3)$ or $(m \geq 3, n \geq 5)$, and if $s_1 > s_2$, then the scoring rule associated with s is not *Maskin-monotonic* (Moulin 1983, attributed to Peleg 1984).[3] This result is a corollary of our Theorem 1. To see this, note that if $s_1 > s_2$, then $k^*(s) = 1$, and under the aforementioned assumptions, we clearly have $k^*(s) = 1 \leq \frac{m(n-1)}{n}$.
2. If $m \geq 3, n \geq 3, (m, n) \neq (3, 4)$, and $n \geq m$, then no scoring rule is *Maskin-monotonic* (Peleg 1984).[4] This result is exactly what our Proposition 1 states. For the sake of completeness, we included a proof.

These results add to our understanding of which scoring rules are not *Maskin-monotonic*. Yet, they do not cover all the possible cases and do not provide a full characterization of *Maskin-monotonic* scoring rules. We fully characterize *Maskin-monotonic* scoring rules and say how many there are of them as a function of the number of alternatives and voters.

Scoring rules are characterized by "symmetry" and "consistency" (Young 1975). Borda's rule is the only rule that is "neutral", "consistent", "faithful", and has the "cancellation property" (Young 1974). The *Maskin-monotonicity* of scoring rules have been studied and in particular "minimal monotonic extensions" (Sen 1995) have been characterized (Erdem and Sanver 2005). The domains of preference profiles on which Borda's rule is *Maskin-monotonic* have also been characterized (Puppe and Tasnádi 2008). *Maskin-monotonic* rules within a special class of scoring rules, the k-plurality rules, have also been characterized (Doğan and Koray 2007).

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[3] The result is Lemma 4 in Chapter 3 of Moulin (1983).

[4] The result is Theorem 2.3.22 of Peleg (1984).